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LXXXIV. *Collisional Friction on Electrons moving in Gases.* By E. C. CHILDS, B.Sc., Ph.D.*

IT is customary to include in the equation of motion of an electron moving in a gas under the influence of electric fields a term representing a force of frictional type, *i.e.*, a resistance to the motion proportional to velocity. The equation of motion may then be written

$$m \frac{d^2x}{dt^2} + g \frac{dx}{dt} = eE_x, \quad . \quad . \quad . \quad . \quad (1)$$

where the electron of mass m and charge e moves in the direction, say x , of the field E_x . This simple expression implies no motion excepting that produced by the field. This is justifiable provided that the velocity acquired, due to the field, between collisions is small compared with the random velocity due to thermal agitation. The coefficient g is the frictional term referred to, and it can be shown † that it is given by

$$g = \frac{m}{\tau} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this expression τ is the average time elapsing between two consecutive collisions, and the value is deduced on the

* Communicated by Prof. E. V. Appleton, F.R.S.

† See, for example, Townsend, 'Electricity in Gases,' p. 84.

assumption that directed momentum is completely destroyed at each impact. A more rigorous treatment gives *

$$g = \frac{m}{0.75\tau}.$$

This is only valid for conditions such that the free paths are distributed according to an exponential law, *i. e.*, they are determined by the laws of chance. This implies that the mean free path must be much smaller than the dimensions of the vessel containing the gases and the electrons. When these dimensions set an effective limit to the lengths of the free paths, the value of g increases, and in particular when all free paths have the same length, *i. e.*, the separation of the walls of the vessel when the gas is at very low pressure, g becomes equal to $\frac{2m}{\tau}$. In the present work

only the first-mentioned conditions are considered.

The frictional forces play an important part in the refraction of wireless waves at the ionized layers in the upper atmosphere, for they are the cause of components of current in phase with the alternating electric fields, and hence determine the absorption. If this is sufficient, the circular polarization of downcoming waves is accounted for †.

It was considered that experimental confirmation of the value of the frictional force calculated on a kinetic theory basis would be of value, particularly as doubts had been expressed ‡ as to the ability of an unmodified kinetic theory to yield correctly the mean free path of an electron, even to the right order. Briefly, Ponte and Rocard have shown, by applying the reasoning of Sutherland § and Wellisch || to electrons, that the electronic mean free path is not $4\sqrt{2}$ times that of the molecules of the gas in which the electron moves, but is only about one twenty-fourth of this, *i. e.*, τ is only about $\frac{1}{160}$ of the commonly accepted value. This is due to the polarization of homopolar molecules or rotation of permanent dipoles by the approaching electron. The field thus set up draws in the electron, possibly to make a collision which otherwise would not occur. The authors then proceed to show that the phenomenon of "skip-distance" indicates that this result is quite incompatible with the

* Loeb, 'Kinetic Theory of Gases,' p. 448.

† Appleton and Ratcliffe, Proc. Roy. Soc. A, cxvii. p. 576 (1928).

‡ 'L'Onde Electrique,' viii. p. 179 (1929).

§ Phil. Mag. (5) xxxvi. p. 507 (1893).

|| Phil. Trans. A, ccix. p. 249 (1909).

experimental results of Appleton * and his co-workers, which yield a Heaviside layer height of only some 90–100 km. In fact this height is an over- rather than an under-estimation, since in the present state of knowledge of the charge distribution of the layer it is impossible to allow for reduction of group velocity suffered by the wireless waves.

The property of ionized gas upon which attention was focussed in the present work was its electrical conductivity when high-frequency alternating electric fields were impressed upon it. A knowledge of this, together with the electron concentration, suffices to determine g †, for

$$\sigma = \frac{Ne^2g}{m^2p^2 + g^2}, \quad \dots \dots \dots (3)$$

where σ is the conductivity of ionized gas containing N carriers per c.c., each of mass m and charge e , when the field frequency is $\frac{p}{2\pi}$. As we have seen, it is necessary

that the mean free path of the electrons should be but a small fraction of the dimensions of the apparatus, this condition necessitating high pressure or, what is the same thing, large g . When this condition is satisfied, g is large compared with mp for all ordinary values of p . (3) then reduces to the simple form

$$\sigma = \frac{Ne^2}{g}. \quad \dots \dots \dots (4)$$

Determination of N and $\frac{m}{\tau}$.

In order to compare the experimental value of g with the theoretical value of $\frac{m}{\tau}$, it is necessary to know the average velocity of the electrons or, what is actually measured, the electron temperature to which the Maxwellian velocity distribution corresponds. In this work, then, full use was made of Langmuir's probe-current analysis method ‡, for it is capable of yielding information both on the electron concentration and on the electron temperature.

The method consists of observation of the current to a probe electrode, inserted in the discharge in the region

* Proc. Roy. Soc. A, cxxvi. p. 542 (1930) and preceding papers.

† Appleton and Childs, Phil. Mag. (7) x. p. 974 (1930).

‡ Gen. Elec. Rev. xxvii. p. 449 (1924).

to be investigated, when the potential with respect to some fixed point, usually the anode, is varied. The probe is conveniently a short length of fine wire, in the present case of tungsten of diameter 0.07 mm. sheathed in glass except for a protruding length of 7 mm. A curve of total current against voltage is plotted, and from the strongly negative region information is obtained which enables the total current to be corrected for positive ion contribution. The logarithm of the electron current, i_- , is then plotted against voltage, and the curve is found to be a straight line over part of the range, of slope

$$S = \frac{d(\log_e i_-)}{dV} = \frac{e}{KT}, \quad \dots \dots (5)$$

K being Boltzmann's constant and T the absolute temperature of the electrons. This expression is only valid for current carried by retarded electrons. At space potential the semi-log curve departs from linearity, and at this point the electron current is given by

$$i_- = ANe \sqrt{\frac{KT}{2\pi m}}, \quad \dots \dots (6)$$

A being the probe area and N the electron concentration in the discharge. N can be calculated from (6) and (5) while \bar{v} , the average velocity of the electrons, can be obtained from (5) for

$$\bar{v} = \sqrt{\frac{8}{\pi} \frac{KT}{m}} \dots \dots (7)$$

A typical semi-log curve is shown in fig. 1. N may also be found from the accelerated electron current, but it was found that such currents were accompanied by distorted discharges, the results being meaningless.

The value of τ is, of course, easily obtained from (7), for

$$\tau = \frac{\lambda_e}{\bar{v}}, \quad \dots \dots (8)$$

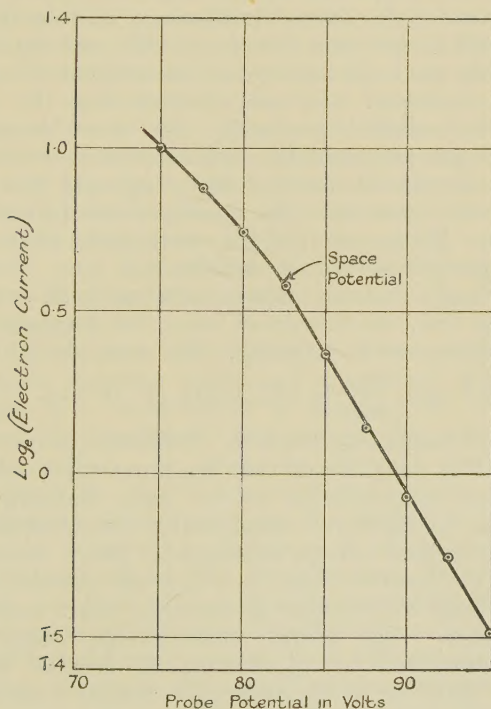
λ_e , the electronic mean free path, being obtained from tables.

Determination of σ .

As we have seen, once N is known it is only necessary to find the electrical conductivity, σ , in order to ascertain the value of g . Many methods were tried before σ was satisfactorily evaluated. In all of them the device of

substitution of resistances of known value for the leak-resistance of the ionized gas was resorted to. In one method the ionized gas, produced by a direct-current discharge between cold electrodes in a cylindrical tube containing air at low pressure, constituted the core of a screened solenoid. Eddy currents in the core had the effect of reducing the effective self-inductance of the coil. In

Fig. 1.



other cases the ionized gas was placed in the field between two parallel condenser plates, forming the whole or part of the dielectric. The difficulty in this case is that, if positive ion sheaths form, results are not capable of numerical interpretation, while if the sheaths are removed by maintaining the condenser plates at space potential, the damping is so excessive that the triode oscillator supplying the high-frequency field ceases to function. Some of the effects of sheath formation had been observed in earlier work on the

dielectric constant of ionized gas *. Some anomalous results showing the inconsistencies produced by sheath formation, will be given.

Two condenser plates were fixed externally to a discharge-tube so mounted that it could easily be removed from the field between the plates and an exactly similar tube containing a mixture of xylol and alcohol substituted. The condenser plates were connected to the ends of a Lecher wire system, which was excited by a triode oscillator oscillating at a frequency of about 3×10^8 cycles per sec. The amplitude of the potential difference between the plates was indicated by a crystal detector circuit, and the damping produced was used as a measure of the resistance in parallel with the condenser. It was assumed that the mixture of xylol and alcohol producing the same damping as the ionized gas possessed the same electrical conductivity. It may be mentioned that the high frequency was used in an endeavour to penetrate the sheaths which formed on the glass walls. Measurement of the conductivity of the liquids at low frequencies presented no difficulties.

It was found that the experimental value of g was very much larger than the calculated value, but decreased as the ionization increased in intensity. For example, with a tube current of 6 m.a. $\frac{g_{\text{obs.}}}{g_{\text{calc.}}} = 136$, while at 10 m.a. $\frac{g_{\text{obs.}}}{g_{\text{calc.}}} = 45$,

gas-pressure remaining constant. This result is not remarkable when it is remembered that the probe experiments give the electron concentration in the main discharge, while positive ion sheaths have a considerably lower concentration. Consequently they have a considerably lower conductivity, to which the positive ions, in which the sheaths are rich, make but little contribution by reason of their considerable relative mass. The overall conductivity, which is the quantity measured by substitution of similar bodies of liquids, is therefore much lower than that of a similar body of main discharge containing the measured number of electrons per c.c. Reference to equation (4) shows that this error produces an excessive value of g . Moreover, as the electron concentration is increased, the sheaths become thinner in consequence of space-charge laws †, so that the discharge becomes more and more nearly uniform over the whole cross-section of the tube. The overall conductivity

* Appleton and Childs, *loc. cit.*

† Child, *Phys. Rev.* xxxii. p. 492 (1911); Langmuir, *Phys. Rev.* ii. p. 450 (1913).

is therefore more nearly equal to that of the core of uniform discharge, and g tends towards the calculated value. Tonks*, working with mercury vapour, recognizes the sheath effect, but neglects it in obtaining a resistance value of the right calculated order. This is probably due to the great mass of the positive mercury ions, for the positive ion current determines sheath thickness, and the space-charge equation shows that, other things being equal, a heavy carrier requires a shorter distance between source and collector.

It was clear from these results that the sheaths must be removed, and this was impossible while they formed on the walls of the tube. Hence, in the next tube built provision was made for the introduction of the condenser plates inside the tube. They were of Eureka, 1 cm. by 1.5 cm. each, and were spot-welded to short Eureka wire helices. The plates were introduced into the 3 cm. diameter tube from one end, and the stiff wire leads, which also were the sole supports, were pushed down capillary side arms and into the wire helices. The latter were slightly too small, so that the leads could only be pushed in by turning against the direction of winding, thus insuring reasonably good contact and rigid support.

These plates were maintained at space potential, and therefore the previous method of comparison with known resistances, *i. e.*, observation of damping of the circuit containing the condenser, was useless, for reasons which have been stated. In order to maintain oscillations it was necessary to couple the oscillator to the condenser *via* a step-down transformer, and the method of comparison used was the observation of change of effective inductance of the primary coil of this transformer. The latter was connected, together with a variable condenser, to a triode to form a source of maintained oscillations, as in fig. 2.

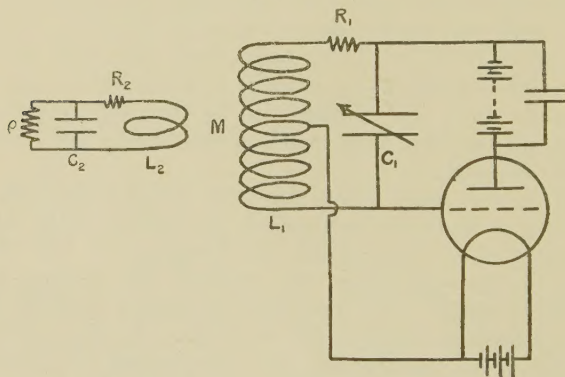
The oscillation frequency was about 10^6 cycles/sec. As is well known, closing of the secondary circuit through resistance produces a decrease of effective self inductance of the primary, this in turn causing an increase in frequency of the oscillations. This change was so small in practice as to be observed only by change of pitch of the heterodyne note produced by the oscillator of fig. 2 and another of slightly different and fixed frequency. This change of pitch was the means of comparing the conductivity of the ionized gas (the leak-resistance ρ of fig. 2) with known liquid conductivities, an audio-frequency oscillator being used

* Tonks, Phys. Rev. xxxvii. p. 1458 (1931).

to measure change of pitch. It is necessary to develop the simple air-cored transformer theory for cases, such as that under consideration, where the reactance of the condenser in the secondary circuit is of the same order as the leak resistance at the frequencies used, and where the primary circuit is tuned.

We will first consider the resonant frequency of the circuit. This is not exactly equal to the natural frequency, but is approximately the same as the frequency of *maintained* oscillations. In fig. 2, C_2 and ρ represent the ionization condenser and its leak (either due to ionized gas or connected resistance) respectively. It is connected to L_2 , a coil of one or two turns of resistance R_2 . The latter is coupled by

Fig. 2.



mutual inductance M to the primary coil, of inductance L_1 , and resistance R_1 , which is tuned by the condenser C_1 . Consider an alternating E.M.F. of form $E_0 \exp(jpt)$ injected into the primary circuit. Then the primary and secondary currents, i_1 and i_2 , may be obtained as follows.

Let the vector resistances of the primary and secondary circuits be written, respectively,

$$\left. \begin{aligned} Z_1 &= a + jb, \\ Z_2 &= c + jd, \end{aligned} \right\} \dots \dots \dots (9)$$

where j represents $(-1)^{\frac{1}{2}}$. Then we have, as the expression of Kirchhoff's laws,

$$\left. \begin{aligned} i_1(a + jb) + jpMi_2 &= E_0 \exp(jpt) = E, \\ i_2(c + jd) + jpMi_1 &= 0. \end{aligned} \right\} \dots \dots (10)$$

Solving for the primary current we find

$${}_0i_1 = \frac{E_0}{\sqrt{\left(a + \frac{cp^2M}{c^2 + d^2}\right)^2 + \left(b - \frac{dp^2M^2}{c^2 + d^2}\right)^2}}. \quad (11)$$

In this expression ${}_0i_1$ is the current amplitude. Reference to fig. 2 shows that

$$\left. \begin{aligned} a &= R_1, \\ b &= pL_1 - \frac{1}{pC_1}, \\ c &= R_2 + \frac{\rho}{1 + p^2C_2^2\rho^2}, \\ d &= pL_2 - \frac{pC_2^2\rho^2}{1 + p^2C_2^2\rho^2}. \end{aligned} \right\} \dots \dots (12)$$

If the circuit is tuned to resonance by variation of L_1 or C_1 , then b is the only variable, and resonance occurs when

$$b = \frac{dp^2M^2}{c^2 + d^2}$$

$$\text{i.e.,} \quad pL_1 - \frac{1}{pC_1} = \frac{dp^2M^2}{c^2 + d^2} \dots \dots (13)$$

The resonant frequency therefore depends on both C_2 and ρ , and the dependence may more clearly be seen by substitution of circuit constants. C_2 is of the order of 0.1 cm., ρ lies between 10,000 and 100,000 ohms, and therefore $p^2C_2^2\rho^2$ is quite negligible compared with unity. R_2 is negligible compared with ρ , being at most a few ohms at radio frequency, therefore,

$$\left. \begin{aligned} c &\doteq \rho, \\ d &\doteq p(L_2 - C_2^2\rho^2), \end{aligned} \right\} \dots \dots (14)$$

and d is much smaller than c . Hence (13) may be approximately rewritten

$$pL_1 - \frac{1}{pC_1} = \frac{dp^2M^2}{c^2} \dots \dots (15)$$

Hence, when ρ decreases then c decreases and d increases, while when C_2 decreases c remains constant and d increases.

Therefore, when either or both of C_2 and ρ decrease

we see from (15) that $pL_1 - \frac{1}{pC_1}$ must increase if the resonant frequency is to remain unaltered, *i. e.*, the value of the tuning capacity, C_1 , must be increased. Alternatively, if C_1 remains constant the resonant frequency increases, and this, as already described, was the method of carrying out the experiment. More laborious substitution, without approximations, yields the same conclusions.

Conversely, increase of C_2 or ρ , or both, leads to a decrease of resonance or natural frequency. C_2 and ρ always vary together and in the same direction, for the dielectric constant of ionized gas is known to be reduced by increasing ionization *. In this connexion it is interesting to note yet further experimental evidence confirming the necessity for sheath removal. When the condenser plates were left free to assume their own potential, sheaths formed, which increased the effective value of C_2 beyond the un-ionized value *, and kept ρ large (resistance of positive ion sheaths). In this case the oscillation frequency was found to decrease, as expected.

Since reduction of ρ is accompanied by reduction of C_2 , and both result in increase of oscillation frequency, it is clearly necessary, in order to measure conductivity, to be able either to separate the effects or to be able to assign to each its proper proportion of the total frequency change. If k is the dielectric constant of ionized gas,

$$\left. \begin{aligned} k &= 1 - \frac{4\pi N e^2 m}{g^2}, \\ \sigma &= \frac{N e^2}{g}, \end{aligned} \right\} \dots \dots \dots (16)$$

so that the larger g is the more nearly is change of C_2 negligible compared with change of ρ . Substitution in (13) showed that if g is of the order of 10^{-17} , change of ρ is almost solely responsible for change of frequency, while with g equal to 10^{-18} the effect of change of ρ outweighed that of change of C_2 by about three to two. It was proposed to determine g approximately by neglecting consideration of change of C_2 , and by successive approximation to arrive at a final value. The method, however, was found to be insufficiently sensitive to make this process worth while. The results, however, showed that the observed value of g was of the same order as the theoretical value.

* Appleton and Childs, *loc. cit.*

Experimental Results.—I.

Air-pressure. = 1 mm. Hg.

Mean free path of electrons (calc.*). = 3.6×10^{-2} cm.

Tube current.	Electron speed, \bar{v} .	Electron concentration, N .	Conductivity, σ .	$g(\text{obs.}) = \frac{Ne^2}{\sigma}$.	$g(\text{calc.}) = \frac{m}{\tau}$.
6 m.a.	6.6 volts = 1.4×10^8 cm./sec.	$9.5 \times 10^7/\text{c.c.}$	2.2×10^{-14} e.m.u.	0.96×10^{-18} dyn./cm./sec.	3.5×10^{-18}
10 m.a.	4 volts = 1.1×10^8 cm./sec.	$2.5 \times 10^8/\text{c.c.}$	4×10^{-14} e.m.u.	1.4×10^{-18} dyn./cm./sec.	2.75×10^{-18}

Efforts were directed towards increase of sensitivity by (a) increasing frequency change, and (b) decrease of local noise from the H.T. generator producing the discharge, but experience showed that progress did not lie in this direction. It may be remarked that the gas conductivity was compared, not with liquids of the same shape and known conductivity, but with known resistances of sputtered type. These all possessed self-capacity comparable with that of the ionization condenser, but different resistances possessed sensibly the same capacity. A 3-megohm leak was therefore connected in parallel with the ionization condenser during the discharge experiments, and was replaced by leaks of lower resistance when comparison was being sought. In this way the self-capacities of the resistances were eliminated. It was proposed to correct for edge-effects by a single experiment with a liquid of known conductivity in the tube. This would, of course, spoil the tube for further work, and neither this nor the successive approximation mentioned was carried out, the accuracy of measurement of σ not warranting the labour. Attention was then turned to another method.

Final Determination of σ .

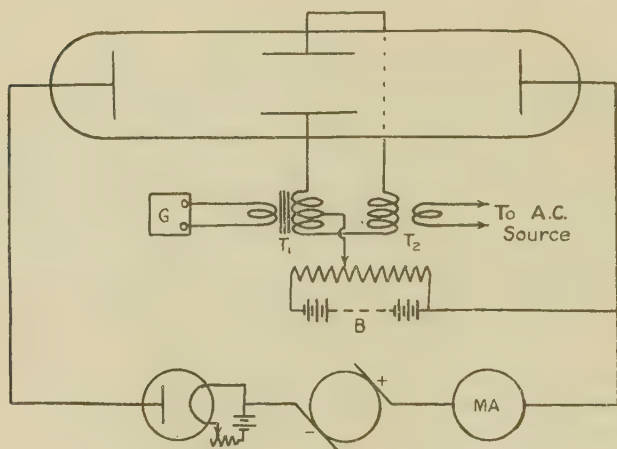
It is not difficult to see that in order to minimize the effect of change of capacity on the impedance of the ionization-condenser circuit the reactance of the condenser

* Mean free path of molecules from Bloch, 'Kinetic Theory of Gases.'

should be very large compared with the equivalent parallel resistance. This implies the use of quite low frequencies, and at once rules out the heterodyne method. The method finally adopted was the comparison of the current passed by the ionization condenser with that passed by various high resistances of sputtered type. The discharge-tube of the previous experiment was retained and the circuit is shown in fig. 3, which is self-explanatory.

Alternating E.M.F.'s of small amplitude were injected into the circuit by the transformer T_2 . Large amplitudes were necessarily avoided, since the potential of the condenser plates must never depart sensibly from space-potential,

Fig. 3.



to which they were adjusted by means of the battery B and its potential divider. When the discharge was struck, in general the potential divider was out of adjustment, and sheaths formed. These were visible as relatively non-luminous regions surrounding the condenser plates. Small alternating currents flowed in the condenser circuit and were detected by the measuring instrument G coupled by the step-down transformer T_1 . G was at first the most sensitive vacuo-junction obtainable (6 mv. for 2 m.a. heater current), but this was abandoned, partly because of lack of sensitivity, but mainly because of violent self-oscillation of the discharge at certain conditions of pressure and tube current. A tuned detector was plainly indicated. A 300 cycle/sec. moving-coil vibration galvanometer was kindly

lent by Messrs. H. Tinsley and Co., Ltd.; this instrument had, of course, greatly superior sensitivity. With this galvanometer the method was immediately successful. The 300 cycles per sec. alternating current in the circuit was detected without a trace of the tube oscillations, at some other frequency, creeping in.

The experimental procedure was as follows. The galvanometer was first tuned to resonance with the frequency of the source, and a reading without discharge taken. This was not zero, owing to a little mutual inductance between the coils of the oscillator and the secondary winding of the transformer T_1 , although the offending units were separated as far as was convenient and suitably orientated. The discharge was then struck, the gas-pressure having first been adjusted to a suitable value and measured by a McLeod gauge.

On striking the discharge, the galvanometer spot opened out slightly, and sheaths were in general observed. The condenser-plate potential was gradually altered so that the sheaths decreased in thickness; at the same time the galvanometer spot was seen to widen slowly. As the condenser plates approached space-potential, the spot rapidly widened, and the reading was taken when the sheaths just disappeared. Owing to the length of the plates (1 cm.) and the potential fall along the positive glow in which they were placed, they were not at space-potential over the whole area at any one instant. The error introduced, however, was slight, the galvanometer deflexion when the sheath over the edge nearer the cathode disappeared being not appreciably different from that when the whole sheath was too thin to be observable. In fact, the effect of the potential fall would not have been noticed had it not been observed that glow due to ionization by collision set in over the cathode edge of the condenser plate and spread as the potential was made more and more positive with respect to space-potential.

These observations having been made for various values of tube current, a series of Langmuir probe experiments were carried out as already described. The discharge was then stopped and various high resistances of known value were connected in parallel with the ionization condenser. From the curve obtained by plotting conductance against vibration galvanometer deflexion, the conductance of the ionized gas was read off. Finally, the edge correction for the condenser was ascertained by filling the tube with alcohol of known conductivity and measuring the conductance of the condenser under these conditions. In this

way the factor for converting conductance of the ionization condenser into conductivity of the medium between the plates was found to be 0.57.

Experimental Results.—II.

Air-pressure = 1 mm. Hg.

Mean free path of electrons (calc.) = 3.6×10^{-2} cm.

Tube current.	Electron concentration, N .	Electron velocity, v .	Conductivity, σ .	$\frac{g \text{ (obs.)}}{= \frac{Ne^2}{\sigma}}$	$\frac{g \text{ (calc.)}}{= \frac{m}{\tau}}$
5 m.a.	$9.6 \times 10^7/\text{c.c.}$	1.3×10^8 cm./sec.	1×10^{-14} e.m.u.	2.5×10^{-18}	3.2×10^{-18}
10 m.a.	$2.4 \times 10^8/\text{c.c.}$	1.5×10^8 cm./sec.	1.9×10^{-14} e.m.u.	3.1×10^{-18}	3.8×10^{-18}
15 m.a.	$3.6 \times 10^8/\text{c.c.}$	1.7×10^8 cm./sec.	3.7×10^{-14} e.m.u.	2.5×10^{-18}	4.3×10^{-18}

These results show that the frictional coefficient of electrons moving in air is not very different from the value calculated from kinetic theory, and is at least of the same order. Hence any calculation of the behaviour of the Heaviside layer based on this assumption is not appreciably incorrect.

Discussion.

It is, perhaps, not surprising that the deductions of Ponte and Rocard * have not been confirmed by experiment. It has been known for some years that the shortening of mean free path due to attraction, while true of molecular ions, cannot be applied to electrons. Methods devised by Ramsauer † and his co-workers and by Jones ‡ for the measurement of electronic mean free paths have demonstrated that these, while departing from simple kinetic theory values, are not of the order of one-hundredth of the latter. Even in the case of the inert gases, which depart most from expected behaviour, the shortening is not large.

* *Loc. cit.*

† Ramsauer, *Phys. Zeit.* xxix. p. 823 (1928); Ramsauer and Kollath, *Ann. der Phys.* (5) iii. p. 535 (1928): iv. p. 91 (1930).

‡ Jones, *Phys. Rev.* (2) xxxii. p. 459 (1928).

and takes place over only a small range of velocities. In the case of nitrogen, which gas undoubtedly predominated in the present experiments, no marked peculiarities were observed by Ramsauer. It is true that the conditions of the experiments in the cases cited were quite different from those of the present work and from conditions prevailing in the Heaviside layer. In the former case well-defined beams of electrons having velocities comprised within a narrow range were used, while in the latter instances the electron velocities are distributed according to the Maxwell law. This case has also been considered by Langmuir and Jones*, who find no more than 10 per cent. discrepancy between observed and kinetic theory values of electronic mean free path, although in this work the electron speeds were somewhat higher than those prevailing in the present investigation.

The assumption of Ponte and Rocard that the molecular polarization is correctly calculable from dielectric constant data is probably at the root of the discrepancy between their theory and experiment. These data are available only for steady electric fields or alternating fields of long time period compared with the time in which a molecule is in the field of an electron travelling with a few volts of energy. The electron velocity was taken into account in determining whether a given polarized molecule would attract the polarizing electron to make a collision, but it was not realized that in dealing with very short time intervals the electric moment is itself a function of the velocity of the electron, and is smaller the faster the approaching electron. In the case of permanent dipoles, the rotation into the direction of the electron is, on account of the moment of inertia, a function of electron velocity. There is some experimental evidence in support of this conjecture, for Ramsauer, working with very slow electrons (less than .1 volt) finds a tendency for some gas-molecules to present very large target areas, *i. e.*, the electronic mean free paths are very short. At normal temperatures, however, there are comparatively few electrons with such velocities.

The author wishes to express his indebtedness to Prof. E. V. Appleton, F.R.S., in whose laboratory at King's College, London, the investigation was made, for help and discussion throughout the work, which was carried out with the aid of a grant from the Radio Research Board of the Department of Scientific and Industrial Research.

* *Phys. Rev.* (2) xxxi. p. 357 (1928).

LXXXV. *Theory of Heat Conduction and Convection from a Low Hot Vertical Plate.* By W. S. KIMBALL and W. J. KING, Michigan State College, East Lansing, Mich., and General Electric Company, Schenectady, N.Y.*

THIS paper is a mathematical theory of heat conduction and convection from a low hot vertical plate in air. Heat transfer by radiation is not considered. It is hoped that exhaustive treatment of this simplest case will open the way for adequate theory of heat transfer from more complicated structures. It is assumed that heat transfer takes place by pure conduction through gas in stream line motion. This assumption restricts the applicability of this theory to low vertical plates not more than about 2 feet high. For experiment shows † that above this limiting height turbulence sets in and the state of affairs is quite different from that described here.

Outstanding recent contributions to the subject are the Langmuir Film Theory ‡ and Rice's § development of it by dimensional analysis. The gist of the film theory is that heat transfer laws etc. are exactly what they would be in case of pure conduction across a stationary film of gas having uniform (in the vertical plate case) thickness determined by the measurable physical constants that apply. The Langmuir Theory deals with resultant effects, and does not purport to include the details of the mechanism of heat transfer across this film; and the associated idea of pure conduction across the included stationary gas is a useful but hypothetical physical picture quite distinct from the actual physical details.

The present treatment, on the other hand, is a detailed analysis based on the actual gas temperature and velocities inside an actual film having for outer boundary an isothermal surface at temperature halfway between the hot surface and the ambient air, and which approaches the equivalent Langmuir film of uniform thickness asymptotically as the height increases.

The experimental basis used to check it is very recent (within three years), and thorough work by E. Schmidt ||, of

* Communicated by the Authors.

† Griffiths and Davis, Food Inv. Bd. Report no. 9 (H.M. Stationery Office).

‡ I. Langmuir, Phys. Rev. xxxiv. p. 401 (1912).

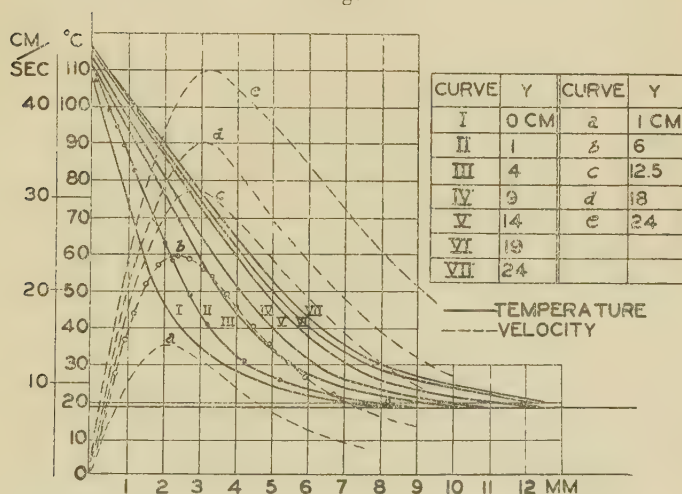
§ C. Rice, Trans. A. I. E. E. xlii. p. 653 (1923); xliii. p. 131 (1924).

|| E. Schmidt, *Zeits. des Gesamte Kalte-Industrie*, p. 213 (Nov. 1928).

Danzig, Germany, and Nusselt and Jürges*, of Dresden, Germany. Schmidt's measurements are given in fig. 1, from which the isotherms of fig. 2 have been plotted. Although Nusselt and Jürges did not measure the velocities directly, their results are in qualitative agreement with those of Schmidt. In fig. 4 the mean temperature opposite the height (H) of the plate is calculated from the measured temperature field by means of the equation

$$t_m = \frac{1}{H} \int_0^H t \, dy.$$

Fig. 1.



Temperature and velocity fields in the air over a heated vertical plate 25 cm. high (from Schmidt).

The mean velocity curve is calculated from a relationship between temperature and velocity, which they derived from theoretical considerations.

In the following analysis the mode of treatment is novel in that it involves, besides the simplified hydrodynamic equations, two approximate empirical laws.

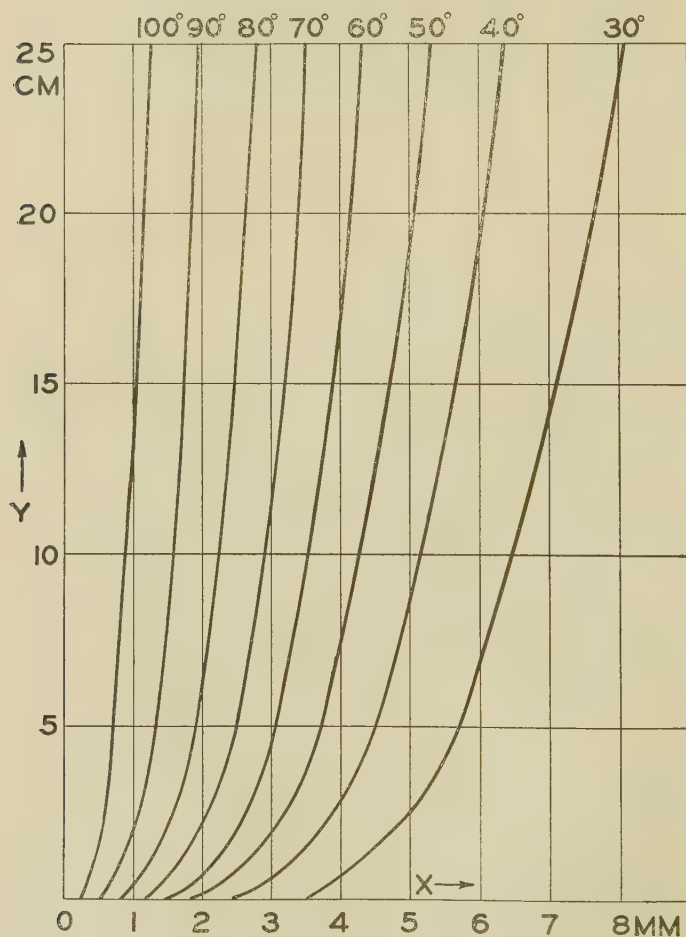
Law I.—The locus of the maxima of the velocity curves is an isothermal surface whose temperature

$$T_m = \frac{T_1 + T_0}{2}$$

* W. Nusselt and W. Jürges, *Zeits. des V. D. I.* lxxii. p. 597 (1928).

is halfway between that of the hot plate and the ambient air.

Fig. 2.

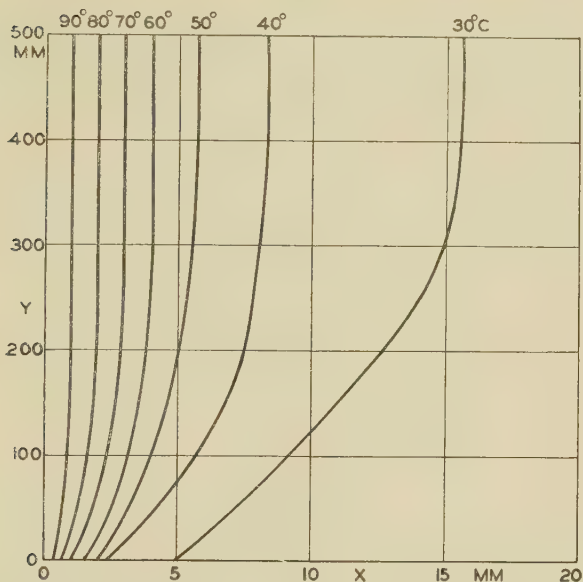


Isotherms in the temperature field over the plate.

It is a striking *experimental* fact (apparently) that this isothermal T_m , halfway between T_1 and T_0 , has on it the maxima of all the velocity curves.

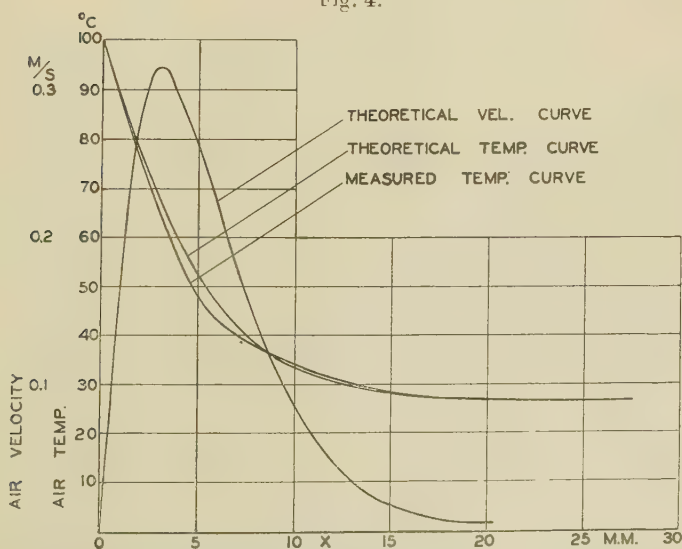
Within this surface the temperature gradient, like that for pure conduction, is constant (for each value of the height

Fig. 3.



Isotherms for a plate 50 cm. high (from Nusselt and Jürges).

Fig. 4.



Mean temperature and velocity curves for the 50 cm. plate (from Nusselt and Jürges).

y), i. e., $T = T_1 - ax(1 + be^{-ay})$ is the empirical expression for temperature inside the surface, where a , b , and α are constants, two of them determined below. Hence the surface is taken as the boundary of the "film" (a modification of the Langmuir film since the present one is of variable thickness).

Law II.—Half the heat is convected up away inside this film and half outside.

The equations for the isothermal surfaces within the film are given by solving the above temperature expression

$$x = \frac{T_1 \cdot T}{a(1 + be^{-ay})}.$$

The equation for the film boundary is the special case of the above, where $T = \frac{T_1 + T_0}{2}$:

$$x_m = \frac{T_1 - T_0}{2a(1 + be^{-ay})} \cdot \cdot \cdot \cdot \cdot \cdot (1)$$

The empirical temperature expression

$$T = T_1 - ax(1 + be^{-ay})$$

satisfies a differential equation,

$$-K \left(\frac{d^2 T}{dx^2} + \frac{d^2 T}{dy^2} \right) = \frac{cdT}{dy}, \cdot \cdot \cdot \cdot \cdot (2)$$

which differs from the classical heat equation for the case in hand *only* in that $c = \alpha K$ (where K is the conductivity coefficient), being approximately constant, instead of $c = \frac{3}{2} knv$

the convection energy flux density per degree which varies from zero at the hot plate to a maximum at the film boundary. It is not yet determined why the classical heat theory, which seems to be applied correctly, does not fit the experimental facts as represented by the above empirical expression for temperature. If we take $K = 27 \cdot 10^2$ for the conductivity coefficient in c.g.s. units (see below, p. 901), and the experimental value of α as between .3 and .08 (see experimental check at the end), then $c = \alpha K$ is a constant between 200 and 1000 c.g.s. units in magnitude; on the other hand, the value

of c called for by the classical theory, $c = \frac{3}{2} knv = \frac{3pv}{2T}$, ranges

from zero at the hot plate to about $28 \cdot 10^4$ c.g.s. units at the film boundary, where $v = 44$ cm./sec. (using $T = 340$ at the

latter place, see table of constants, p. 901). The experimental value of c is seen to be a fair average (as regards order of magnitude) between the extremes of zero and $28 \cdot 10^4$ of the classical heat equation; but the fact that it is constant rather than variable can scarcely be accounted for by turbulence, since the following analysis is based on stream-line conditions and affords excellent experimental checks. Rather this constancy of $c = \alpha K$ seems to indicate the need for some modification of the classical theory as applied to heat transfer within the film.

The fundamental equations of hydrodynamics are simplified by neglecting *within* the film inertia effects, second order velocity effects, and horizontal velocities, and using the gas law. One force equation is then left:—

$$-\eta \frac{\partial^2 v}{\partial x^2} = n_0 mg - nmg = mg(n_0 - n). \quad (3)$$

The right member is the buoyancy (molecular mass times the acceleration of gravity is multiplied by the difference in molecular concentration at temperature T and at ambient temperature T_0). The left member is the viscous drag (coefficient of viscosity multiplied by the second derivative of the velocity) *.

THEOREM I.—*The rate of doing work per unit volume by the buoyancy is proportional to the rate of heat transfer through unit area.*

If we use the gas law $p = nkT$ (p is the pressure, n the molecular concentration, k Boltzmann's constant, and T the absolute temperature) the above equation can be written

$$-\eta \frac{\partial^2 v}{\partial x^2} = mg(n_0 - n) = \frac{mgp}{kT_0} \frac{T - T_0}{T}. \quad (3)$$

If v is the velocity of the convection currents the rate of doing work by the buoyancy force is the product of v by the above force equation:

$$\frac{mgp}{kT_0} \left(\frac{T - T_0}{T} \right) v = \text{rate of work per unit volume by the buoyancy} \quad (4)$$

(Compare this rate with the rate of heat convection through unit area. This comprises the excess of molecular energy of a molecule where T is the temperature over the molecular

* Lehigh Page, 'Introduction to Theoretical Physics,' p. 230.

energy for the ambient temperature $\frac{3}{2}k(T-T_0)$ multiplied by nv , the concentration times the velocity of flow :

$$\frac{3}{2}k(T-T_0)nv = \frac{3}{2}p \frac{(T-T_0)v}{T} \quad . \quad . \quad . \quad (5)$$

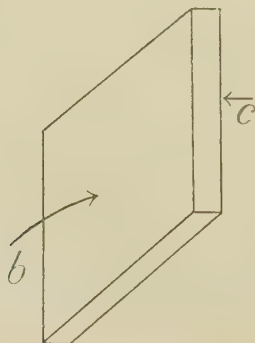
This is seen to be proportional to the rate per unit volume of doing work by the buoyant forces, and about a million times larger (proportionality factor is $\frac{mg}{\frac{3}{2}kT_0}$).

It is noteworthy that the buoyant forces are the *sole* forces that push the gas upwards in convection currents, being opposed by viscous drag and inertia effects (gravity is included in the buoyancy). If the inertia effects are negligible, then, according to (3) and Theorem I., the rate of doing work against viscosity is proportional to the rate of heat transfer. This is in accord with the famous proportionality of Osborne Reynolds between fluid friction and heat transfer, which is thus where (3) and (5) apply put on an exact basis. This proportionality, however, between viscous drag and heat transfer is seen not to hold where inertia effects prevent the validity of (3). On the other hand, the rate of doing work by the buoyancy (4) is always proportional to the rate of heat transfer (5) wherever the gas law and stream line conditions prevail.

THEOREM II.—*The viscous forces operating in gas in a steady state form a mechanical couple, and hence their sum is always zero. Viscosity's sole mechanical effect is that of a couple: it passes on the reactions from place to place within the gas or to the walls of the container, together with a corresponding torque.*

Consider a vertical slice of unit cross-section and thickness dx :

Fig. 5.



$$-\eta \frac{dv}{dx_b} = \text{viscous drag down at the left.}$$

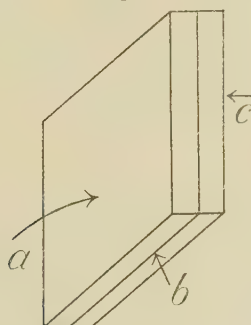
$$\eta \frac{dv}{dx_c} = \text{viscous drag up at the right.}$$

$$\eta \left(\frac{dv}{dx_c} - \frac{dv}{dx_b} \right) = \frac{d}{dx} \eta \left(\frac{dv}{dx} \right) dx = \eta \frac{d^2v}{dx^2} dx$$

is the excess force down at the left over that up at the right, being the resultant downward drag that balances the buoyancy, $mg(n-n)dx$, of the slice according to (3). If $\frac{d^2v}{dx^2}$ were zero, the viscous drag would have no effect on balancing buoyancy *within* the gas.

Now consider two similar slices together, each of thickness dx and unit area :

Fig. 6.



$$-\eta \frac{dv}{dx_a} = \text{viscous drag down at left of first slice.}$$

$$\eta \frac{dv}{dx_b} = \text{viscous drag up at right of first slice.}$$

$$-\eta \frac{dv}{dx_b} = \text{viscous drag down at left of second slice.}$$

$$\eta \frac{dv}{dx_c} = \text{viscous drag up at right of second slice.}$$

Evidently the resultant effect of the viscous drag between the slices on surface b is zero, due to action and reaction. Likewise other slices may be considered on each side of those depicted, and the total effect is solely to pass along the reactions to the hot plate at the left of the film. The

summation of these effects between the hot plate and film boundary gives

$$-\int_0^{x_m} \eta \frac{d^2 v}{dx^2} dx = \eta \frac{dv}{dx_1} = mg (\overline{n_0 - n}) x_m. \quad (6)$$

Here the integral vanishes at the upper limit, where the slope of the velocity curves is zero, and the right member is readily understood to be the total buoyancy (for unit cross-section) between hot plate and film boundary, and $\overline{n_0 - n}$ is the mean value of the concentration difference. The middle term is the viscous upward pull per unit area on the hot plate, being equal in magnitude and direction to the total buoyancy. Thus the buoyancy is not really balanced by viscosity, but its effect is merely transferred from within the gas to the wall of the hot plate where the equivalent mechanical upward thrust operates without diminution. Viewed as a couple $\eta \left(\frac{dv}{dx_1} \right)$ is the upward viscous force (positive) at the hot plate, and

$$\int_0^{x_m} \eta \frac{d^2 v}{dx^2} dx$$

is the sum of the downward reactions (negative) throughout the gas, being equal and opposite to the former; and besides this there is, of course, the corresponding torque.

Consider now the effect of viscosity outside the film. At the point of inflexion of the velocity curve the second derivative is zero and since the buoyancy is obviously different from zero we see that equation (3) does not apply, and we must add an inertia term:

$$mg (\overline{n_0 - n}) = -\eta \frac{d^2 v}{dx^2} + mnv \frac{dv}{dx} \quad (7)$$

Assuming this to be the vertical force equation, integrate between film boundary x_m and the point of inflexion x_f :

$$mg (\overline{n_0 - n}) (x_f - x_m) = -\eta \frac{dv}{dx_f} + mnv \frac{dv}{dx} (x_f - x_m). \quad (8)$$

The left member is the total buoyancy per unit cross-section and the right member is the balancing downward pull of

viscosity plus inertia working together in the same direction. Now integrate (7) between the limits infinity and x_f :

$$mg(\overline{n_0 - n})(\infty - x_f) = \eta \frac{dv}{dx_f} + mnv \frac{dv}{dx} (\infty - x_f); \quad (9)$$

$$\text{or} \quad mg(\overline{n_0 - n})(\infty - x_f) - \eta \frac{dv}{dx_f} = mnv \frac{dv}{dx} (\infty - x_f). \quad (10)$$

Between these limits $\frac{d^2v}{dx^2}$ is positive, and so the viscous drag per unit volume is up rather than down. That is to say, viscosity is pulling up over this interval and is helping the buoyancy to react against the now large inertia term. Thus each term in the left member of (10) is now positive ($\frac{dv}{dx_f}$ is minus) and represents upward force, whereas the right member corresponds to the downward inertia reaction. Now add equations (8) and (9), cancelling out viscosity:

$$mg(n_0 - n)(x_f - x_m) + mg(\overline{n_0 - n})(\infty - x_f) = mnv \frac{dv}{dx} (x_f - x_m) + mnv \frac{dv}{dx} (\infty - x_f). \quad (11)$$

The left member is the total buoyancy between film boundary and infinity and the right is the total balancing inertia, and the equation shows by cancellation of vertical viscous forces how viscosity contributes nothing, according to Theorem II., towards balancing buoyancy except torque. Its sole effect is to pass on the extra buoyancy of (8), where the gas is hot, to help pull up the cold gas where (9) and (10) apply. Viewed as a mechanical couple the downward viscous forces represented by the viscous term of (8) are equal and opposite to the upward viscous forces beyond the point of inflexion x_f represented by the viscous term of (9), the former being balanced by buoyancy and the latter by inertia. Although this theorem is illustrated by differential equations referring to a special case, the reasoning is general and can probably be extended to other cases, and hence the theorem may be one of general application.

Corollary—The total buoyancy is always balanced by inertia effects plus a downward pull by the walls of the container equal and opposite to the viscous drag on them.

Application of the Laws.

The integration of the force equation, expanding in series and using second order terms, and using Law I. to determine the constant of integration ($\frac{dv}{dx} = 0$ when $x = x_m$ according to (1)), gives

$$v = B(x^3 - Cx^2 + Dx) \left\{ \begin{array}{l} \text{where } B = \frac{pmga(1 + be^{-\alpha y})}{6k\eta T_1^2}, \\ \text{and } C = \frac{6T_1}{T_0} x_m, \\ \text{and } D = x_m^2 \frac{12T_1}{T_0} \left(\frac{1 - T_0}{4T_1} \right). \end{array} \right. \quad (12)$$

The maximum value of v for changes in x is (substitute (1))

$$v_m = \frac{pmg(T_1 - T_0)^3}{12k\eta T_0 T_1 a^2 (1 + be^{-\alpha y})^2} \quad (13)$$

$$\left(\text{taking } \frac{1}{2} - \frac{T_0}{6T_1} = \frac{1}{3}, \text{ for } T_0 = 293, T_1 = 383 \right).$$

The heat conduction per unit area at the hot plate is

$$q = -K \frac{\partial T}{\partial x} = Ka(1 + be^{-\alpha y}).$$

$q = h(T_1 - T_0)$ where K is the conductivity coefficient, h the heat transfer coefficient, and a , b , and α are constants; a and α are expressed below in terms of physical constants, whereas b is purely experimental as yet.

The total heat convected away (per unit plate width) is then

$$Q = \int_0^L q dy = K \int_0^L a(1 + be^{-\alpha y}) dy.$$

$$Q = Ka(L + \frac{b}{\alpha}(1 - e^{-\alpha L})).$$

We take, according to Law II.,

$$KaL = Ka \frac{b}{\alpha} (1 - e^{-\alpha L}), \quad (14)$$

since KaL is the heat transferred to beyond the film, and $Ka \frac{b}{\alpha} (1 - e^{-\alpha L})$ is the excess supply of heat at lower part due

to larger temperature gradient down there, being convected up through the film. The theoretical justification for (14) is tied up with the theoretical incompleteness of (2), involving convection up through the film. Equation (14), like (2), is thus to be taken as an experimental fact, checked below by theory and experiment. We may, however, interpret the equivalence of KaL and $\frac{Q}{2}$ (which is (14)) by reliance on the

Langmuir film concept, that actual rates of heat transfer, heat transfer coefficients, etc. are exactly what they would be if the heat were transported away from a hot body by pure conduction across a film of practically stationary gas, and of thickness determined by the conditions of the problem. For our hot plate we have thus

$$Q = KL \frac{(T_1 - T_0)}{d} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

And since Q , K , L and the temperatures are actual measurable physical quantities, the film thickness d , so determined by them, is a very real and useful concept, being for the case in hand the space interval that would correspond to the entire temperature drop between hot plate and ambient air,

so as to give $T_1 - T_0 = \frac{dT}{dx}$, the right magnitude needed to

take care of Q according to the above equation. The physical picture of the hypothetical state of affairs within the (very actual) film thickness d is that of pure conduction,

i.e., $\frac{dT}{dx}$ is constant hence $\frac{d^2T}{dx^2} = 0$, and hence no loss of heat

occurs *en route* in process of transmission of heat by conduction from hot plate to film boundary. Hence the left member of (15) represents (according to the hypothetical physical picture) heat transferred across the film to the gas *beyond* it and disposed of there. The fact that only part of Q is actually transported to the region beyond and outside the film does not impair, of course, the usefulness of the hypothetical physical picture, with its associated actual value of d , the film thickness determined by (15).

Now divide (15) by 2, retaining this hypothetical physical picture of pure conduction across a film of thickness d :

$$\frac{Q}{2} = \frac{KL}{d} \left(\frac{T_1 - T_0}{2} \right) \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This shows that if half the total temperature drop took place

between hot plate and Langmuir film boundary $x=d$, then in case of pure conduction half the total heat would be transported beyond the film. Conversely, if we have $\frac{Q}{2}$ units of heat transported by pure conduction beyond a plane $x=\text{const.}$, where the temperature drop from the hot plate is half the total temperature drop, then that surface is the Langmuir film boundary $x=d$. Now this latter situation just fits the case in hand, for experiment shows by Law II.

that $\frac{Q}{2}$ units of heat are carried off outside a plane surface $x = \frac{T_1 - T_0}{2a}$ (our film thickness at the top), which, when substituted for d in (16), gives KaL for (16)'s right member, being the heat transported by pure conduction beyond $x=d$, having half the total temperature drop.

Thus $x = d = \frac{T_1 - T_0}{2a}$ is the Langmuir film boundary as checked below by both theory and experiment. Thus heat transferred to the gas from the hot plate has a dual aspect: (a) KaL (the right member of (16)) represents pure conduction beyond the Langmuir film $x = \frac{T_1 - T_0}{2a}$, whereas

(b) $Ka \frac{b}{a} (1 - e^{-aL})$ is convected up away within this plane surface $x = \frac{T_1 - T_0}{2a}$. Therefore

$Q = 2KaL$ is the total input of heat.

Half the output is, by Law II.,

$$\frac{3}{2} \int_0^{x_m} k(T - T_0) n v dx = \frac{1}{28} \frac{p^2 m g (T_1 - T_0)^5}{k \eta T_0 T_1^2 a^3 (1 + b e^{-aL})^3} = \frac{Q}{2} \text{ at } y = L;$$

$$\therefore KaL = \frac{1}{28} \frac{p^2 m g (T_1 - T_0)^5}{k \eta T_0 T_1^2 a^3 (1 + b e^{-aL})^3}.$$

$$\text{Solving for } a = \left(\frac{p^2 m g (T_1 - T_0)}{28 k \eta T_0 T_1^2 (1 + b e^{-aL})^3 KL} \right)^{1/4} (T_1 - T_0).$$

Therefore

$$Q = 2KaL = \left(\frac{16 p^2 m g (T_1 - T_0)}{28 k \eta T_0 T_1^2 (1 + b e^{-aL})^3} \right)^{1/4} (KL) (T_1 - T_0).$$

. . . (17)

This checks the $5/4$ power law for variation of heat transferred with temperature difference $(T_1 - T_0)$.

Since $q = Ka(1 + be^{-ay}) = h(T_1 - T_0)$, we have for the heat transfer coefficient

$$h = \frac{Ka}{T_1 - T_0}(1 + be^{-ay}) = K^{3/4} \left(\frac{\rho^2 mg(T_1 - T_0)}{28k\eta T_0 T_1^2 (1 + be^{-aL})^3 L} \right)^{1/4} (1 + be^{-ay}),$$

which seems to check known data.

Using the above value of a gives for the maximum convection velocity at the top of the plate, where $y = L$,

$$v_m = \frac{5}{12} \sqrt{\frac{mg(T_1 - T_0)KL}{k\eta T_0(1 + be^{-aL})}}, \quad \dots \quad (18)$$

which checks the experimental data of Schmidt. If we use the above value of a in the first derivative of v given by (12), taking $x = 0$, we get the slope of the velocity curves at the hot plate:

$$\frac{dv}{dx_1} = .93 \left(\frac{mg(T_1 - T_0)(1 + be^{-aL})}{k\eta T_0} \right)^{3/4} \sqrt{\frac{p}{T_1(1 + be^{-ay})}} (KL)^{1/4} \quad (19)$$

(here $1 - \frac{T_0}{4T_1} = .87$).

Note that this velocity slope at the hot plate, and hence the vertical viscous drag per unit area there, varies, among other factors, as the three-quarters power of the temperature difference between the hot plate and ambient air.

Equations (18) and (19) may be used as a check on the validity of equation (3) as applied to the case in hand, from which they are derived by integration. Using c.g.s. units, we have for the case of air at atmospheric pressure

$$\begin{aligned} p &= 1.013 \cdot 10^6, & k &= 1.37 \cdot 10^{-16}, & \eta &= 2.00 \cdot 10^{-4}, \\ m &= 46.5 \cdot 10^{-24}, & T_1 &= 383, & T_0 &= 293, \\ g &= 980, & K &= 2272 \sqrt{\frac{383}{273}} = 2720, & a &= 91, \end{aligned}$$

$T_1 - T_0 = 90$, and taking $b = 2$; $(1 + be^{-b}) = 1.278$ (see below). Substituting these constants in (18) gives for v_m , the maximum convection velocity at the top of the plate, a value of about 46 cm./sec. This is almost too good a check on the 44 cm./sec. shown in Schmidt's graph. For $b = 5$, v_m comes to the value 51 cm./sec. It will readily be understood that neglect of inertia forces in equation (3) within the film

accounts for the slightly increased values of v_m , as we shall show in more detail below, and the fact that v_m is so close to 44 cm./sec. is an excellent check on the present theory.

Now substitute these constants in (19), or, perhaps easier in the first derivative of (12), taking $a=91$, and obtain for the slope of the velocity curve at the top of the hot plate a value of about 450 cm./sec./cm. We may compare this with the average slope of the velocity curve within the film obtained roughly by dividing 44 cm./sec. by the thickness of the film .33 cm., *i. e.*, 133 cm./sec./cm. It will be seen by examination of Schmidt's graphs that they indicate to the extent justified by the data that the slope at the hot plate is for the upper curve three or four times the average slope 133, being hence not so far from the 450 given by the present theory. Since the present theoretical values of

v_m , $\frac{dv}{dx_1}$ and $\frac{dv}{dx}$ average, agree relative to each other as well

as in order of magnitude with the corresponding experimental quantities obtained from Schmidt's graphs, we infer that the shape of the theoretical curves matches the shape of the experimental curves, and that the differential equation (3) from which the former spring is thoroughly

justified. Incidentally it is to be noted that $\frac{dv}{dx_1} = 450$ when

multiplied by the coefficient of viscosity gives the viscous drag per unit area at the hot plate. Hence the experimental and theoretical values of this force check one another. Using the above value of η gives this force as about .09 dynes per sq. cm.

More detailed checks are obtained by substituting the above physical constants in equation (3) in the form

$$\eta \frac{d^2v}{dx^2} = \frac{pmg}{kT_1^2} \left(a(1 + be^{-ax})x - \frac{T_1}{T_0} (T_1 - T_0) \right) = mg(n - n_0), \quad (3)$$

which was integrated to get (12). Taking $x=0$ gives the buoyancy per unit volume at the hot plate, *i. e.*, $\eta \frac{d^2v}{dx_1^2} = .27$

dynes/c.c., and since *there* the convection velocity is zero and hence the inertia is zero, this is exactly equal to the viscous drag per unit volume there. To get the buoyancy per unit volume at the film boundary, where $x=x_m$, we substitute (1)

in (3) above, and taking $\left(1 - \frac{T_0}{2T_1}\right) = \frac{17}{27}$ and using the

previous physical constants, we get $\eta \frac{d^2v}{dr_m^2} = .17$ dynes per c.c.

The average of .22 for these buoyancies when multiplied by (.4+), the film thickness, is seen to give approximately the above .09, the viscous upward drag at the hot plate according to Theorem II., which is thus checked numerically.

We may compare the buoyancy per unit volume at the film boundary with the inertia at the same place. Here obviously, from the graphs, this latter is greater than it is nearer the hot plate. Since a steady state prevails, time does not appear explicitly in the inertia terms, and they are all negligible for the case under consideration, except perhaps the type

$$I_m = nmv \frac{dv}{dy} = \frac{pmv}{kT_m} \frac{dv}{dy}, \quad . \quad . \quad . \quad (20)$$

where the subscript m is used to indicate that the temperature refers to the film boundary. Taking $T_m = 350$, and from

the graph $v = 44$ and $\frac{dv}{dy} = \frac{4}{3}$, and using the above physical

constants, gives $I_m = .059$ dynes per c.c. This *largest* value of the inertia reaction is seen to be about one-third of .17, the *smallest* buoyancy term which is at the same place, which we have set by (3) equal to the viscosity. This shows that the inertia term, though small at the top of the plate, is not negligible, and indicates why our maximum velocity of 46 to 51 cm./sec. exceeded the experimental one of 44 cm./sec. If the inertia reactions were balancing one-third the buoyancy *everywhere* between hot plate and film boundary the viscosity would have only 2/3 of the buoyancy left to balance, so that the buoyancy member of (3) would need to be multiplied by 2/3 to give correct results. Since, however, the buoyancy member of (3) needs only to be multiplied by about 11/12 to check the experimental v_m (11/12 of 48 equals 44), it appears that only 1/12 of the total buoyancy forces between hot plate and film boundary is balanced by the corresponding inertia reactions. These numerical conclusions refer of course to the top of the plate which supplied the data, and thoroughly justify the use of (3) in this region.

These considerations show how the method of successive approximations may be used if a more accurate expression for v than (12) is needed, *i. e.*, integrate (7) instead of (3), substituting (12) and its derivative in the inertia term corresponding to (20). Furthermore, we may check the

applicability of (7) beyond the film at the point of inflexion, where $\frac{d^2v}{dx^2} = 0$, and hence the viscous drag per c.c. is zero, and the buoyancy must balance the inertia. If we eliminate the n 's by means of $p = nkT = n_0 kT_0$, and factor out $\frac{mp}{kT}$ from both sides of (7) (without the viscosity term), there remains

$$g \frac{(T - T_0)}{T_0} = v \frac{dv}{dy}.$$

Now refer to Schmidt's graphs and take $v = 25$ and $T - T_0 = 15$ at the point of inflexion of the upper velocity curve. Substituting above gives $\frac{15g}{T_0} = 49 = 25 \frac{dv}{dy}$, which requires $\frac{dv}{dy}$ to be about 2, a value which checks the graph fairly well.

Using the above a in the expression x_m for film thickness gives

$$x_m = \frac{T_1 - T_0}{2a(1 + be^{-\alpha y})} = \left(\frac{28 k \eta T_0 T_1^2 (1 + be^{-\alpha L})^3 KL}{16 \rho^2 mg (T_1 - T_0)} \right)^{1/4} \frac{1}{(1 + be^{-\alpha y})}.$$

This checks the data for variation of film thickness with temperature difference (inversely as the quarter power).

We may express Q in terms of the film thickness x_m (see equation (1)) at the top of the plate where $y = L$:

$$Q = 2KaL = \frac{KL(T_1 - T_0)}{x_m(1 + be^{-\alpha L})},$$

which may be written:

$$Q = \frac{KL(T_1 - T_0)}{d},$$

where $d = x_m(1 + be^{-\alpha L})$ = thickness of the equivalent Langmuir film, *i. e.*, the equivalent film of constant thickness.

This shows that $2a = \frac{T_1 - T_0}{d}$. Therefore, a is one-half the temperature gradient of the equivalent Langmuir film.

The above results check the numerical experimental data.

$$a = \frac{106}{(1 + be^{-b})}$$

= 91 if $b = 2$ and = 102 if $b = 5$ (using the approximation

$b = \alpha L$, which is (14) when the exponential term is neglected). Referring to Schmidt's paper,

$$\begin{aligned} T_1 &= 383, \\ T_0 &= 293, \\ T_1 - T_0 &= 90; \end{aligned}$$

$$\therefore x_m = \frac{T_1 - T_0}{2\alpha(1 + be^{-\alpha y})},$$

taking $b = 2$;

$$\begin{aligned} \therefore x_m &= \frac{45}{91(1 + 2e^{-.08y})}, \\ x_m &= .165 \text{ at } y = 0, \\ &= .38 \text{ at } y = L, \\ &= .49 \text{ at } y = \infty; \end{aligned}$$

this latter is the thickness of the equivalent Langmuir film.

Taking $b = 5$,

$$\begin{aligned} x_m &= \frac{45}{102(1 + 5e^{-.2y})}, \\ x_m &= .077 \text{ at } y = 0, \\ x_m &= .44 \text{ at } y = L = 25, \\ x_m &= .44 \text{ at } y = \infty, \end{aligned}$$

thickness of the equivalent Langmuir film.

The Theory checks

I. The numerical values of α .

II. The $5/4$ power law $Q \propto (T_1 - T_0)^{1/4}$.

III. The maximum velocity is independent of pressure and proportional to $(T_1 - T_0)^{1/2}$.

IV. Film thickness varies inversely as fourth root of temperature difference

$$x_m \propto \frac{1}{(T_1 - T_0)^{1/4}}.$$

V. Heat transfer coefficients are proportional to the square root of the pressure

$$h \propto \sqrt{p}.$$

VI. The thickness of the equivalent Langmuir film (uniform thickness) is

$$d = \frac{T_1 - T_0}{2\alpha} = \left(\frac{28 \cdot k\eta T_0 T_1^2 (1 + be^{-b})^3 KL}{16p^2 mg (T_1 - T_0)} \right)^{1/4}$$

= between .40 and .50 cms. (depending on choice of b).

VII. Law II. is checked by graphical integration to the extent justified by the data and method.

Taking the upper temperature and velocity curves shown in Schmidt's graph as applicable to the top of the plate where $y=L$, the mean values for each square shown in the graph are tabulated :—

No. of Square.	Absolute Temp. $T=t+273.$	Temp. Diff. $T-T_0=t-18.5.$	Veloc. $v.$	Temp. Ratio. $\frac{T-T_0}{T}$	Product. $\frac{T-T_0}{T} v.$	Sum. $\left(\frac{T-T_0}{T}\right) v$	
	1	383	91.5	12	.239	2.86	
	2	370	78.5	30	.212	6.35	
	3	356	64.5	42	.181	7.60	
1/5 interval	4	348	56.5	44.5	.162	1.44	
						} = 18.25.	
4/5 interval	4	343	51.5	44	.150		5.28
	5	331	39.5	40	.119		4.75
	6	321	29.5	36	.092		3.32
	7	312	20.5	31	.066	2.04	
	8	305	13.5	27	.0445	1.20	
	9	302	10.5	23	.0348	.80	
	10	299	7.5	18	.0250	.45	
	11	297	5.5	16	.0185	.296	
	12	296	4.5	13	.0152	.198	
		295	3.5	10	.0117	.117	
						} = 18.45.	

The sums in the right-hand column, when multiplied by $\frac{3}{2}p$, as per equation (5), give the energy flux inside and outside the film respectively, and are seen to be approximately equal.

NOTE.—Since the above paper was written the results of some new experimental and theoretical work on Free Convection have been published by Schmidt and Beckmann*.

Their new temperature and velocity curves are quite similar to those of fig. 1.

The writers are much indebted to Dr. H. Poritsky, of the General Electric Company, Schenectady, N.Y., for valuable constructive criticism.

* E. Schmidt and W. Beckmann, *Tech. Mech. und Thermo.* i. no. 10, p. 341, and no. 11, p. 391 (1930).

LXXXVI. *A Preliminary Report on the Anomalous Variation of the Electrical Conductivity of Quartz with Temperature.*
By SADAKICHI SHIMIZU, *Physical Institute of Tohoku Imperial University, Sendai* *.

§ 1. *Introduction.*

THE variation of the electrical conductivity of dielectrics with temperature has been studied by many experimenters. Curie † has measured the conductivity of quartz plates, cut parallel and perpendicular to their optical axes with the temperature in a range from 7° C. to 300° C., and deduced the formula $\sigma = at^{-n}$, where σ is the conductivity, t the temperature, and a and n the constants.

Tegetmeier and Warburge ‡ have measured the electrical conductivity with a temperature up to 300° C., and reported that the axial conductivity has an electrolytic character.

Exner § has measured the conductivity of quartz in a temperature range 100–150° C.

Ambrohn ¶ has measured the current through the quartz plate with a temperature from 166° C.–466° C., and reduced the expression of the resistivity to

$$R = Ae^{\frac{B}{T-C}},$$

where A , B , C are constants and T the absolute temperature, and has given numerical values also, *i. e.*, the constant C is equal to 224.9 ± 2.3 , and B lies between 1413–1586, and A has different values depending on the direction of the current with respect to the optical axis.

Joffé ¶ has measured the variation of the electric conductivity for quartz, calcite, ammonium alum, and copper-sulphate, and has found the following formula :

$$\log \sigma = \frac{A}{T} + B, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A is the nearly constant for these crystals. Assuming

* Communicated by Prof. H. Saegusa.

† J. Curie, *Compt. Rend. Paris*, ciii. pp. 923–931 (1886), and *Ann. Chem. et Phys.* (6) xviii. pp. 203–269 (1889).

‡ F. Tegetmeier and E. Warburge, *Ann. der Phys. u. Chem.* (Wiedemann) xxxii. pp. 442–451 (1887).

§ Exner, *Verh. Deut. Phys. Ges.* iii. pp. 26–30 (1901).

¶ R. Ambrohn, *Dissert. Göttingen* (1913).

¶ A. Joffé, *Ann. d. Phys.* Bd. lxxii. p. 495 (1923).

the variation of the conductivity with the temperature as due to the dissociation of ions, he gets the expression of A as a function of the gitter constants of these crystals.

Doelter* has measured the resistivity of quartz in the range of the temperature 500°C. – 1400°C. , using the Wheatstone bridge, but no satisfactory result is obtained, especially in the temperature range 600°C. – 900°C.

Recently many experimenters have made a study of the electric conductivity in relation to the temperature effect of some dielectric crystals as a part of the general investigation of conduction mechanisms; for example, N. Ussataja and B. Hochberg† have measured the electrical conductivities of dielectric crystals with temperature, such as NaNO_3 from 20°C. to 360°C. , CaF_2 from 200°C. to 500°C. , NaCl and mica from 100°C. to 500°C. , and deduced the following formulæ:

$$\log \sigma = \frac{A}{T} + B \text{ for } 550^{\circ}\text{C.} - 600^{\circ}\text{C.}, \quad . \quad . \quad . \quad (1')$$

$$\log \sigma = A_1 T + B_1 \text{ above } 650^{\circ}\text{C.}, \quad . \quad . \quad . \quad (2)$$

where A , A_1 , and B , B_1 are constants.

V. Seelen‡ has measured the conduction current of rock-salt with a temperature from 15°C. to 500°C. , and found that the expression of conductivity can be expressed by equation (1).

B. Hochberg and A. Walther§ have measured also the variation of conductivity of rock-salt with a temperature from 25°C. – 800°C. , and concluded that above a temperature 550°C. the Cl ions become the carriers of the electric charge.

W. Seith|| has examined the conductivities of PbCl_2 and PbI_2 , and deduced the following expressions:

$$\sigma = A e^{-\frac{Q}{RT}} \text{ for } \text{PbCl}_2, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\sigma = A_1 e^{-\frac{Q_1}{RT}} + A_2 e^{-\frac{Q_2}{RT}} \text{ for } \text{PbI}_2, \quad . \quad . \quad . \quad (4)$$

* Doelter, *Sitzungs. Kals. Wiss. Wien. Math. Nat. Kl.* cxvii. 1, pp. 845–874 (1908); cxix. 1, pp. 49, 11 (1910).

† N. Ussataja and B. Hochberg, *ZS. f. Phys.* xlv. p. 88 (1927).

‡ V. Seelen, *ZS. f. Phys.* xxix. p. 125 (1924).

§ B. Hochberg and A. Walther, *ZS. f. Phys.* lxiv. p. 39 (1930).

|| W. Seith, *ZS. f. Phys.* lvi. p. 802 (1929).

where A , A_1 , A_2 , and Q , Q_1 , Q_2 are constants, and conclude that in the case of PbCl_2 its anion may be the carrier of the electric charge, and in the case of PbI_2 both ions were taken into account in considering the electrolytic conduction, and the latter case is verified theoretically by Smekal*.

B. Hochberg and A. Walther† have also verified the fact that the conductivity for NaNO_3 , SiO_2 , Ag-halogenides satisfy the expression (3) for PbCl_2 , and Na, K-halogenides satisfy expression (4) for PbI_2 . On the other hand, Königsbergers and Reichenheim‡ have studied the conduction in dielectrics, as the conduction is performed by electrons instead of ions, and deduced the following formula :

$$\sigma = AT^2 e^{-\frac{B}{T}}; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and also H. Saegusa§ deduced theoretically the following expression, assuming that the electric conduction in a dielectric is performed by electrons :

$$\sigma = A T e^{-\frac{\phi}{kT}} \{ 1 - B T^2 e^{-\frac{\phi}{kT}} - C T e^{-\frac{\phi}{kT}} \}, \quad . \quad . \quad (6)$$

where

$$A = \frac{lce^2}{4k} A', \quad B = 2\rho^2 A', \quad C = \frac{2\rho e^2}{k} A',$$

k = Boltzmann constant, ϕ = work function,

l = mean free path, A , A' , B , C = constants.

ρ = density,

From the above mentioned results we cannot find any discontinuous variation of electrical conductivity of dielectrics with respect to temperature, and especially for the α - β transformation temperature of quartz. In the present investigation, we have studied the variation of electrical conductivity of quartz with temperature in a somewhat wider range than those of the above-mentioned physicists, and have considered especially whether any discontinuous variation of conductivity occurs or not at the transformation temperature.

* A. Smekal, *ZS. f. Phys.* lviii. p. 322 (1929).

† B. Hochberg and A. Walther, *l. c.* (9).

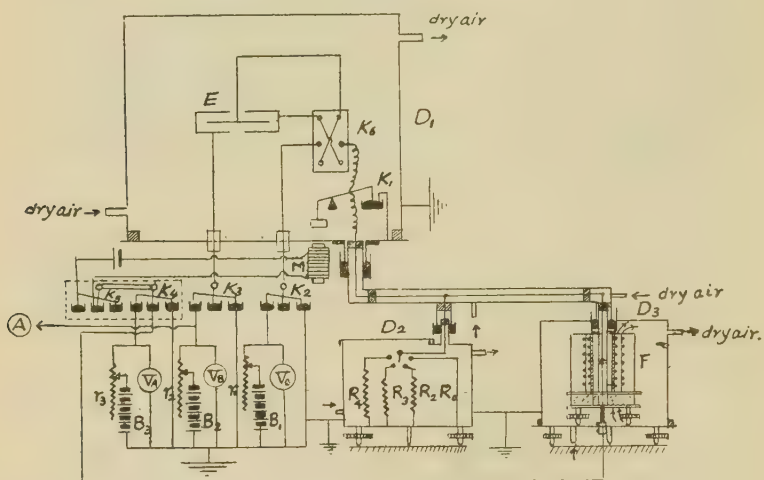
‡ Königsbergers and Reichenheim, *Phys. ZS.* vii. p. 570 (1906); viii. p. 833 (1907).

§ H. Saegusa, 'Sci. Report of the Tohoku Imperial University,' I. vol. xv. no. 6, p. 795 (1926).

§ 2. Apparatus and the Methods of the Experiments.

The main part of the apparatus of the experiment is shown in fig. 1.

Fig. 1.



The symbols in the figure are as follows :—

E, Dolezalek quadrant electrometer.

K₁, K₂, K₃, K₄, K₅, K₆, keys.

M, electromagnet to cut the circuit from the earth.

V_A, precision voltmeter for measuring the applied voltage.

V_B, precision millivoltmeter for measuring the potential of one pair of the quadrants.

V_C, precision voltmeter for measuring the needle potential or the quadrant potential.

r₁, r₂, r₃, resistances to reduce the battery voltage.

R₀, R₂, R₃, R₄, standard resistances for producing the potential fall of the current from the specimen to the earth.

F, electric furnace including the specimen.

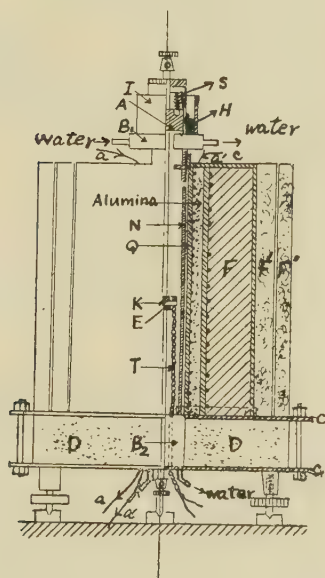
D₁, D₂, D₃, metallic boxes enclosing the apparatus.

As is seen from this figure, the leading wire from the specimen to the electrometer is enclosed entirely with metallic boxes to prevent any external influences. The dry air which passes through the pure sulphuric acid and P₂O₅ is forced into the metallic boxes D₁, D₂, D₃, and P₂O₅ and

H_2SO_4 are placed in the spare spaces of the boxes, and the equilibrium state is kept with a vapour pressure of P_2O_5 .

The variation of the zero point of the electrometer which comes from the geometrical irregularities of the needle due to the temperature difference from day to night is prevented

Fig. 2.



The symbols in the figure are as follows :—

A, supporter of the leading wire made of ambroid.

B₁, B₂, water-cooling tubes made of copper.

C, concentric copper cylinder to prevent external influences.

D, fireproof brick to prevent the dissipation of heat inside the furnace.

E, nickel electrodes supporting the specimen.

H, mercury, which serves for the electrical coupling.

I, iron case.

K, quartz test specimen.

N, concentric cylinder made of nickel to keep the temperature of its interior uniform.

S, spring.

T, thermo-junction made of Pt and Pt-rhodium.

by maintaining constant temperature outside of the box D_1 by using a thermostat made of wood and nichrome wire. The constancy of the temperature is such that its variations is about 1 degree within 10 hours. K_6 is made of ambroid and ebonite with particular precautions, and its insulation is maintained as well as the other supports of the circuit; thus the leakage of the charge from this switch is prevented as much as is possible. When the accumulating charge is small we connect the specimen with one pair of quadrants, and when it is large we connect the specimen with the needle of the electrometer. In the latter case interaction between the applied potential and the measuring charge is prevented completely.

To get rid of the induction effect due to the measuring potential of the specimen a special switch is used, *i. e.*, the switches K_4 , K_5 are coupled together, and it consists of the mercury column and the spring; by operating K_4 , K_5 instantaneously after the measuring potential is given to the specimen, the electrometer side is momentarily earthed and then disconnected from the earth.

When the battery which serves for the magnet M varies its potential the force which moves the switch K_1 also varies, so that we take care especially concerning the battery potential.

When the temperature increases we measure the potential difference of standard resistances R_0 , R_1 , . . . R_4 due to the current from the specimen to the earth passing through them, using a dial switch made of ambroid and ebonite with special precautions.

The electric furnace heating the specimen is shown in fig. 2.

The electrodes and the guard ring of the specimen are platinized by spluttering, under the same conditions. After spluttering both sides of the specimen are earthed more than 200 days, and the effect of previously applied potential of spluttering was removed completely. The leading wire is a very thin nickel wire, because small conducting area prevents the effect of the heat. Since the mixed phases of α and β quartzes exist at 573°C ., and therefore the temperature gradient in the specimen produces a twin, especially during cooling, we diminish the electrode mass, and spend a sufficiently long time to reach the equilibrium state. The earth of the guard ring is composed of three nickel springs, and one part of a thermo-junction is also used for the earthing. The electrodes are made of nickel, which are

supported by a silica tube, and the silica tube is also supported by the ambroid column A. The specimen is

Fig. 3.

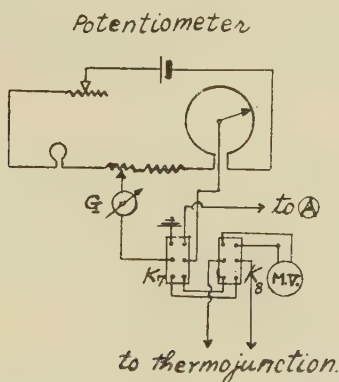
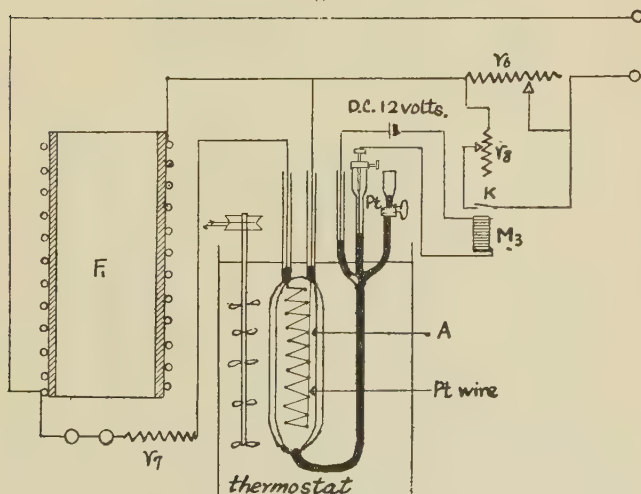


Fig. 4.

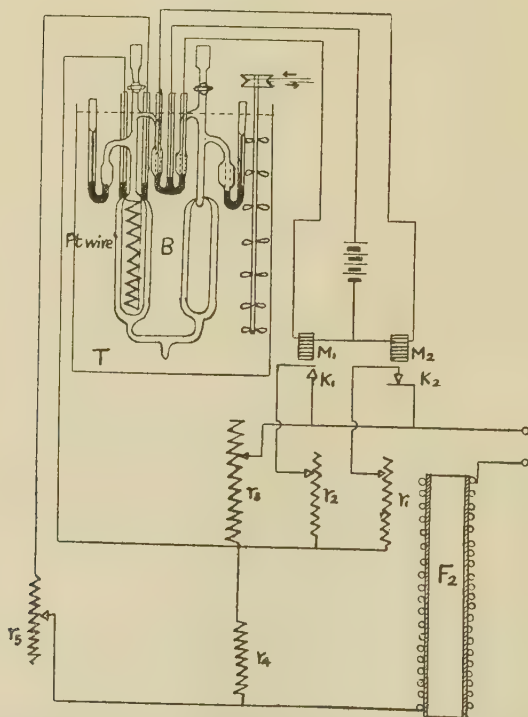


A, toluene regulator; F_1 , outer furnace; K, key; M_3 , magnet;
 r_6 , r_7 , r_8 , suitable resistances.

directly in contact with the electrodes by the spring S, and by this device any piezo-electric effect produced by the mechanical expansion due to temperature increase is avoided.

To avoid the heating of the insulators for the conducting wires from the specimen to the electrometer, and also from the battery to the specimen, the copper rings B_1 , B_2 are used, and cooling water is forced to flow through the insides of these rings, and the upper part I of the ring B_1 is covered with an enamelled iron tube, and the insulator of the leading wire is protected from the moisture and also the spring S is

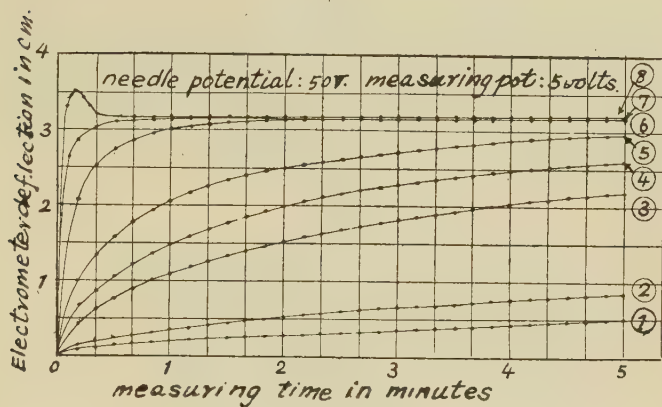
Fig. 5.



B , toluene regulator; F_2 , inner furnace; r_1, r_2, r_3, r_4, r_5 , suitable resistances; K_1, K_2 , keys; M_1, M_2 , magnets; T , thermostat.

protected from rust. To obtain a uniform temperature and to protect from any external effect a concentric cylinder N made of nickel is inserted around the specimen. To keep any required temperature constant we use an electric furnace with double layers, *i. e.*, the inner layer is a quartz tube furnace and the outer layer is a porcelain tube furnace,

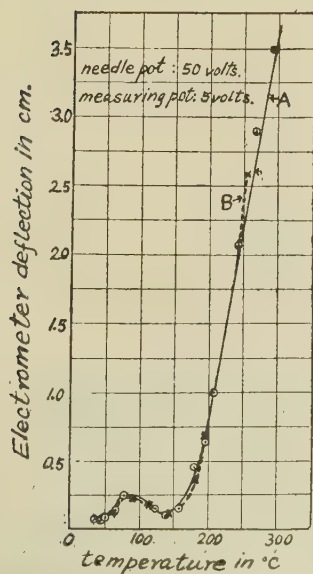
Fig. 6.



Temperature.

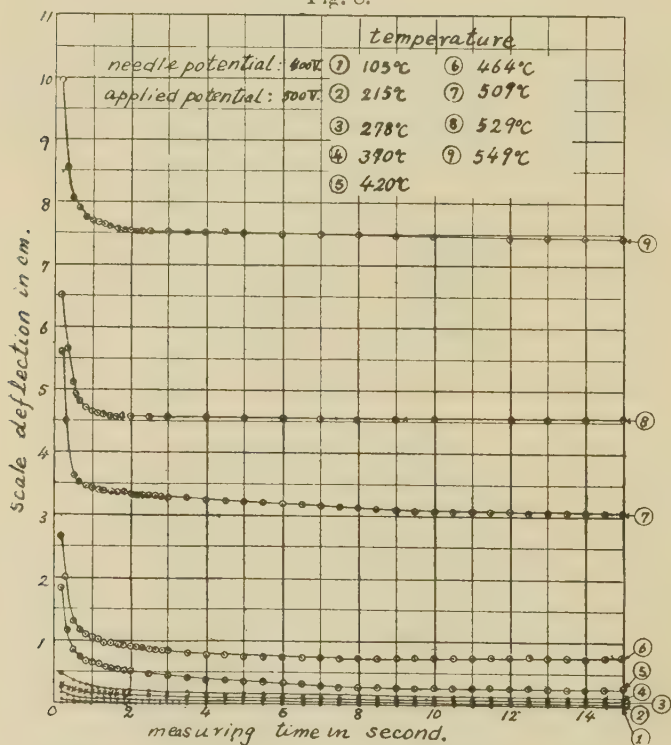
- | | |
|---------------|---------------|
| (1) 136° C. | (5) 208·0° C. |
| (2) 151·5° C. | (6) 240·0° C. |
| (3) 178·0° C. | (7) 267·0° C. |
| (4) 195·0° C. | (8) 297·0° C. |

Fig. 7.



as is shown in fig. 2, and any fine adjustment of temperature is made by the inner layer. F' , F'' are asbestos wool layers which are separated from each other and from F by the concentric copper cylinder C . These protective layers increase the heat capacity of the apparatus and prevent the dissipation of heat. D is the fireproof brick, and the coolers B_1 , B_2 make the temperature gradient in the furnace symmetrical.

Fig. 8.

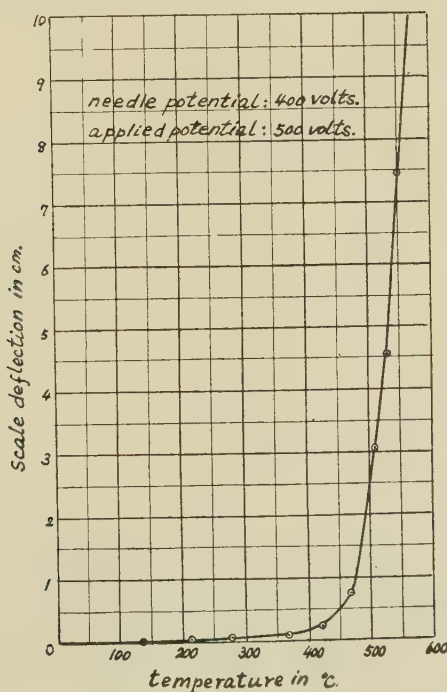


For measuring the temperature of the furnace accurately a thermo-junction made of Pt-Pt rhodium and a millivoltmeter are used, and this millivoltmeter is calibrated by the potentiometer. Sometimes we used the potentiometer to measure the potential difference of the thermo-junction accurately. The main connexion is shown in fig. 3. G is the galvanometer of Leeds and Northrup (current sensibility 10^{-8} amp.), MV the millivoltmeter, and K_7 , K_8

reversing switches. To avoid the variation of voltage due to the temperature difference at contact points of mercury and keys petroleum is used.

Fig. 4 shows the device to maintain the constant temperature of the outer furnace. Fig. 5 is the arrangement of the fine adjustment of temperature in the interior furnace. The relay B in fig. 5 was first made by T. Watase*, and we

Fig. 9.



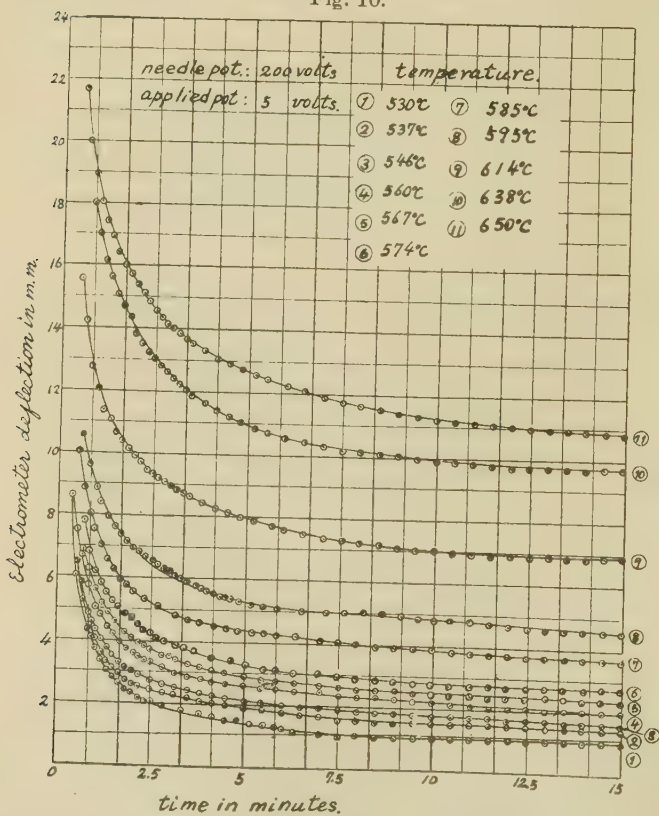
modified his apparatus, *i. e.*, the part which contains toluene was made of double bottoms and their internal parts silvered. The heat produced by the variation of the terminal potential of furnace is maintained during a few minutes; after that an equilibrium state with the surrounding temperature of thermostat is obtained. Fig. 4 is the simple case of fig. 5, and these relays are put into the thermostat. The temperature of the thermostat is adjusted approximately by a gas

* T. Watase, *Kinzoku no Kenkyu*, vol. v. no. 12, p. 1 (1928).

regulator, and, moreover, a small adjustment is made by the electric regulator, and its constancy is such that the variation of the temperature is within $\frac{7}{1000}^{\circ}\text{C}$.

The observation of the accumulating charge is made in

Fig. 10.



quite a similar way to that of the former paper*, i. e., the variation of electrometer deflexion with the time is observed, and then the resistance is calculated from the formula

$$R = \frac{V}{\left(\frac{dQ}{dt}\right)_{t=0}}.$$

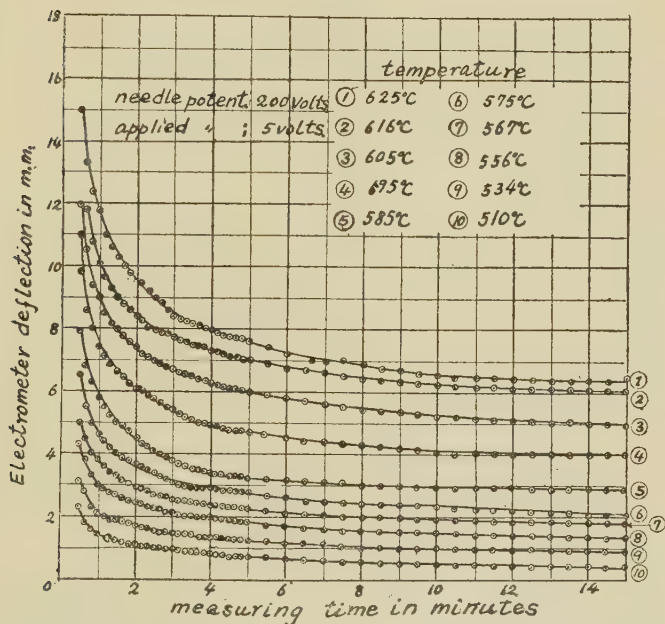
* H. Saegusa and S. Shimizu, Phil. Mag. ix. p. 474 (1930).

For high temperatures, since the accumulating charge becomes rather large, the potential difference due to conduction through the standard resistances is measured, and the apparent resistivity of the specimen is calculated by Ohm's law.

§ 3. Results of Experiments.

(1) The variation with the temperature of the electrical conductivity of a quartz plate perpendicular to axis.

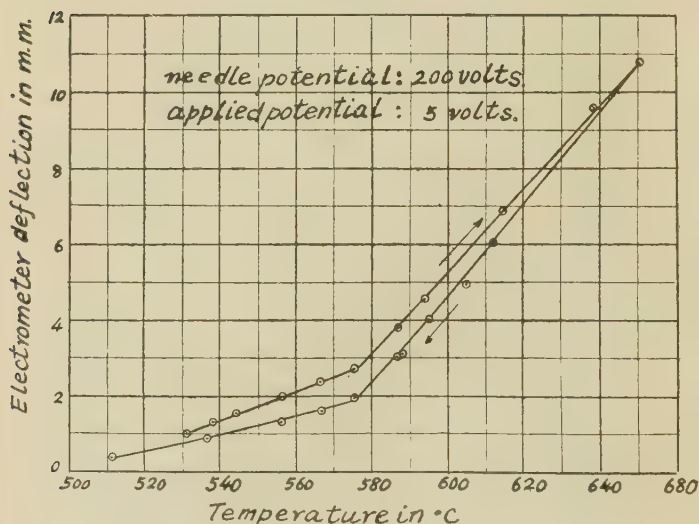
Fig. 11.



* The thickness of the specimen is 0.294 mm., and the electrode area of spluttering is 0.603 sq. cm. Fig. 6 shows the relation between the electrometer deflexion and the time for various temperatures, when the applied potential is 5 volts. Fig. 7 shows the variation of the electrometer deflexion at 10 seconds from the start with the temperature. This shows the variation of the conductivity with the temperature, and curves A, B are measured under the same conditions, and these values coincide with each other. As is seen from this figure the deflexion increases at first slowly, then decreases

a little, and from about 150°C . increases rapidly with the temperature. This phenomenon is probably due to the leakage of the charge through the surface, so that the results given below are measured by the potential difference due to the conduction through a standard resistance. Fig. 8 shows the variation of the potential difference between the ends of the standard resistance, with the measuring time for various temperatures. As is seen from this figure the potential difference decreases at first rapidly and then tends to a stationary value. This shows that the current through the

Fig. 12.



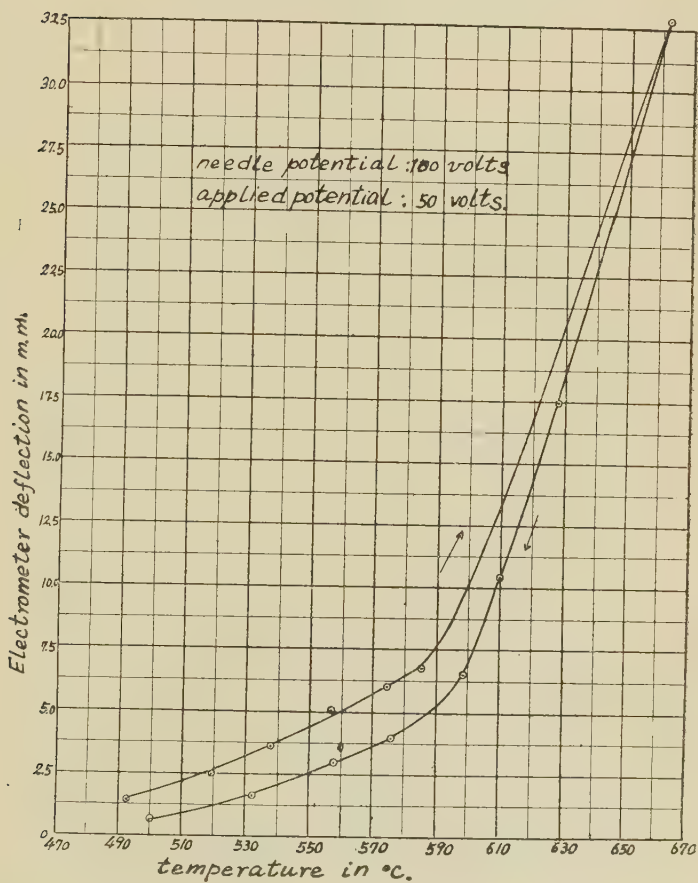
specimen decreases as the opposing potential due to the polarization increases, and tends to a stationary value as the opposing potential tends to a saturation value and the current becomes conduction alone. Fig. 9 shows the relation between the temperature and the stationary value of the potential difference. As is seen from this figure the conductivity increases with the temperature, and from about 400°C . it increases abruptly.

(2) Variation of the conductivity at the transformation temperature.

Fig. 10 is the result of the observation for the measuring potential of 5 volts and the needle potential, 200

volts for the increasing temperatures; the temperatures are given in the figures. Fig. 11 is the same relation for decreasing temperatures. To study the variation of conductivity about the transformation point accurately we use

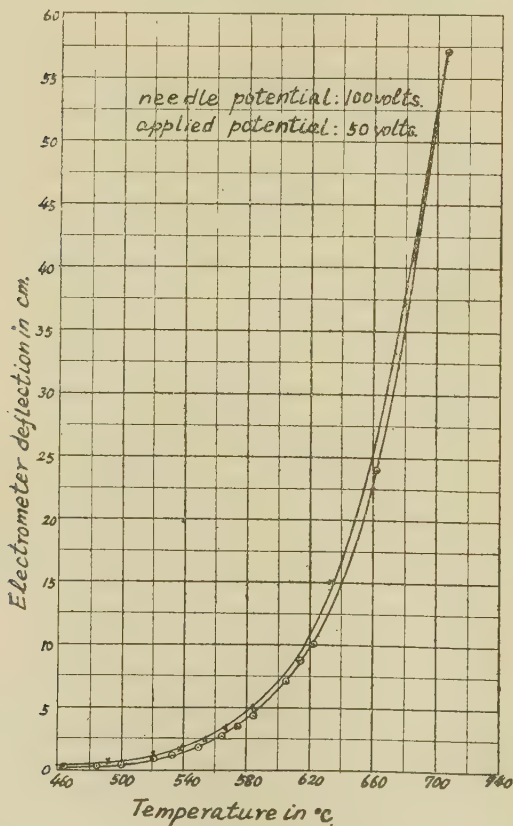
Fig. 13.



the standard resistance 800,000 ohms as an auxiliary, and observe the deflexion of the electrometer carefully with temperatures from about 500° to 600° C., the increase and decrease of temperature being made as slowly as possible, *i. e.*, the time interval is about 15 hours. Fig. 12 shows the relation between the stationary value of electrometer deflexion and the temperature taken from figs. 10 and 11. As is seen from

this figure the electrometer deflexion (*i. e.*, the conductivity) increases linearly with the temperature, and at 573°C . it increases abruptly and transforms to another linear relation. The arrow indicates the increasing and decreasing temperature. Thus the conductivity for a decreasing temperature is a little smaller than that for an increasing one.

Fig. 14.



(3) Change of the variation of conductivity at the transformation point due to the effects of electrode's oxidation.

We used nickel plates as the supporting electrodes of the specimen; hence the electrodes are oxidized at the high temperature. The oxidation of the electrodes affects apparently the conductivity of the specimen. Fig. 13 is the result of

a measurement after maintaining at the temperature about 500°C. during 40 hours. As is seen from this figure the conductivity gradually increases up to about 580°C. , then increases rapidly with the temperature. Then the same specimen is kept still further for about 40 hours at the temperature of 500°C. , and the observation is made. Fig. 14 is the result thus obtained. As is seen from this

Fig. 15.

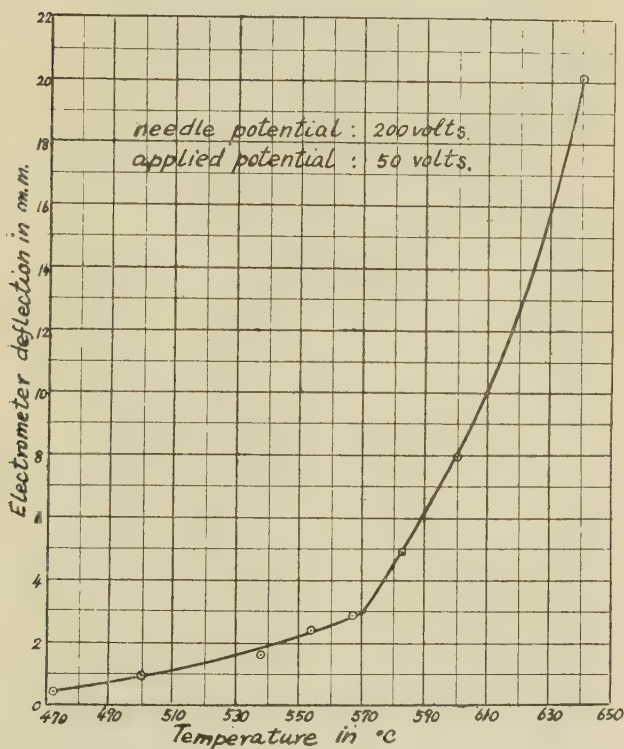
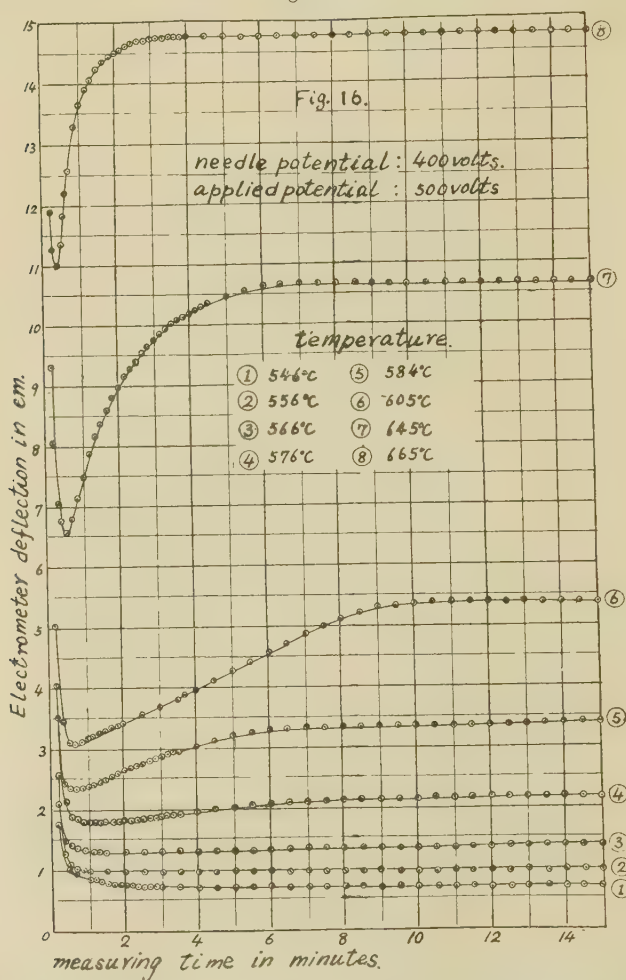


figure the relation between the conductivity and the temperature at 573°C. becomes a continuous curve. Next we polish the electrode surfaces and make a good contact with a platinized quartz plate, and a similar measurement is again made. In this case the increase of conductivity at the transformation point becomes somewhat sharp, as is shown in fig. 15. Thus we may conclude that the oxidation of the

electrode modifies the abrupt increase of conductivity at the transformation point. As is seen from the above results it is a remarkable fact that the electric conductivity increases

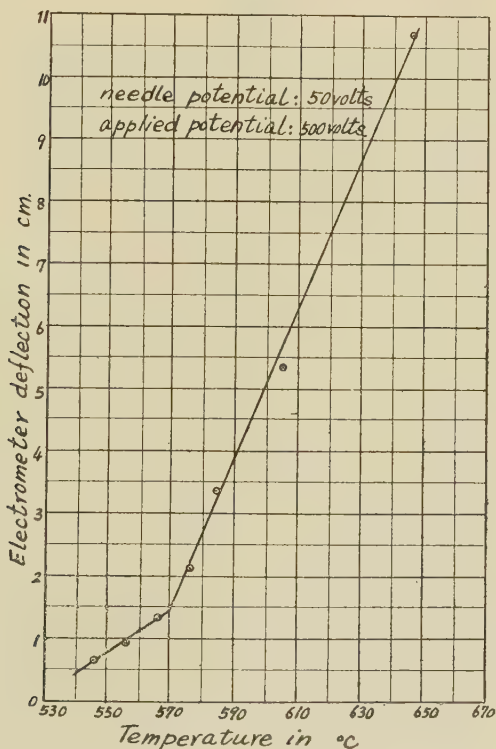
Fig. 16.



abruptly at the transformation point, and the transformation point is sharply defined by the conductivity, and the oxidation of the electrode modifies the transformation point. Many experimenters who have examined the conductivity of quartz platinized or silvered the specimen, but electrode metals were

very different; *e.g.*, R. Ambrohn* used silvered copper electrodes, Exner† used liquid electrodes, Tegetmeier and Warburg‡ used platinum electrodes, and Doelter§ also used the platinum electrodes. In order to get definite values we must use the same electrode metal and be careful concerning its oxidation.

Fig. 17.



(4) Conductivity for a large measuring potential.

Next we measured the conductivity for a large measuring potential above the limiting potential. Fig. 16 shows the result for a measuring potential of 500 volts. As seen from this figure the conductivity varies as that of the semi-conductors. Fig. 17 shows the variation of the stationary

* *Loc. cit.*

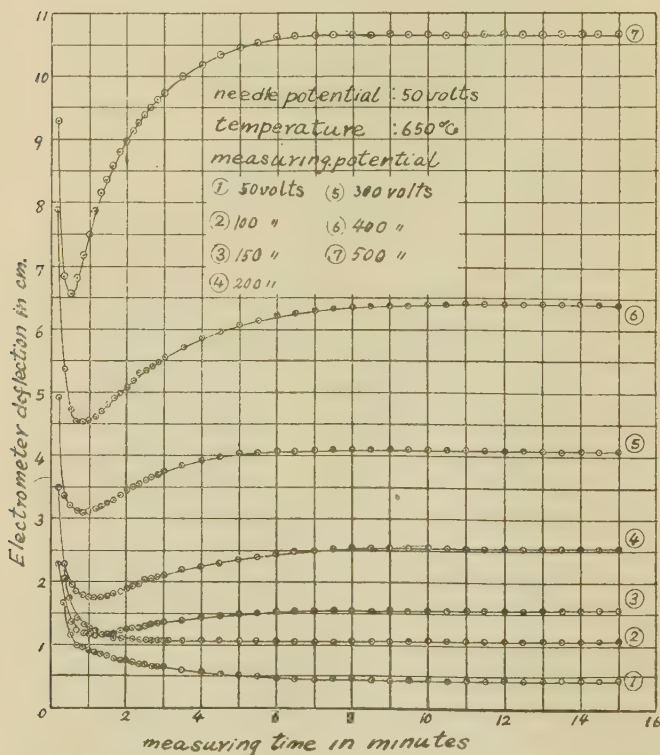
† *Loc. cit.*

‡ *Loc. cit.*

§ *Loc. cit.*

value of potential difference with the temperature. In this case also the conductivity increases discontinuously at the temperature 573°C . To study the initial decrease of potential difference as in fig. 16 we measured it with various measuring potentials for the case when the temperature is 650°C . These experimental results are given in figs. 18 and 19; the former is for positive potential and the latter for negative

Fig. 18.

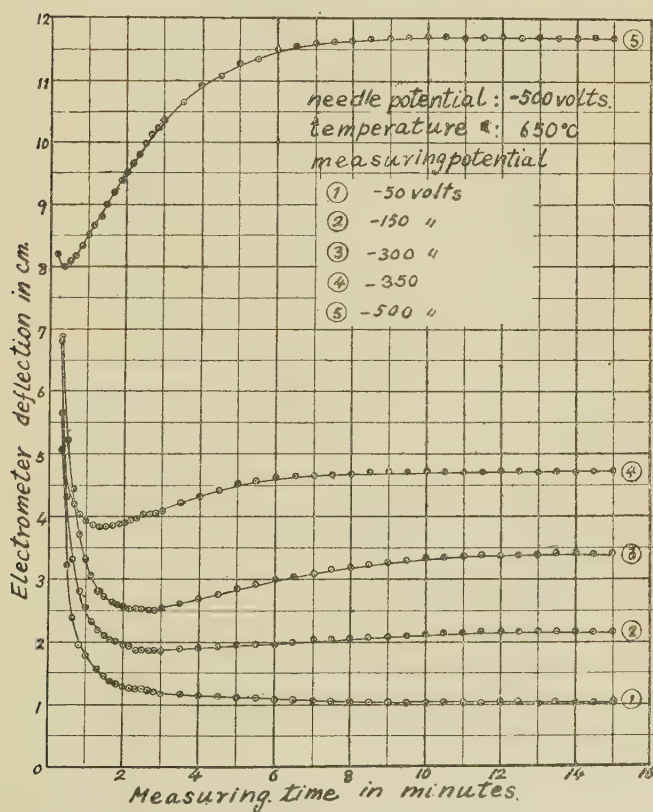


potential. As is seen from these figures the initial decrease appears for large measuring potentials; thus we state that dielectrics for a large measuring potential at high temperature become as if they were semi-conductors. Fig 20 shows the relation between the stationary value of conductivity and the measuring potentials obtained from figs. 18 and 19, *i. e.*, the A curve is for the positive potential and the B curve for the negative potential. The A curve shows a sharp part

above the limiting potential for dielectric hysteresis, but the B curve does not; moreover these two values are very different. On these different characters we should like to state no definite conclusion for the present.

(5) Variation with temperature of electrical conductivity of quartz plate parallel to axis.

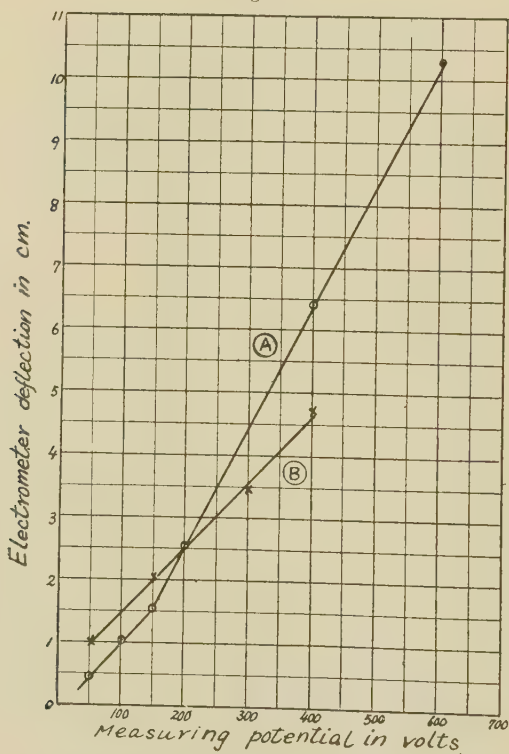
Fig. 19.



We studied this problem with a quartz plate cut parallel to its optical axis. The specimens used in the experiment are two plates, one of them 0.477 mm. in thickness and the other 0.480 mm. thick, and these plates are taken from the same parent crystal, and for convenience we call the former the A specimen and the latter the B specimen. For the

parallel plate the conductivity is comparatively smaller than that of the perpendicular specimen, so that we used the accumulation-charge method in this case. Fig. 21 shows the relation between the electrometer deflexion and the temperature, when the needle and the applied potential are both 50 volts, where A, B curves are for A, B specimen respectively, and these values are nearly equal. Fig. 22 shows the

Fig. 20.



result when the needle potential is 50 volts and the applied potential is 5 volts. As is seen from this figure the variation of conductivity at the temperature 573°C . shows the discontinuous change as in the case of quartz plate cut perpendicular to its optical axis, but the variations are somewhat different.

Fig. 23 shows the relation between the electrometer deflexion and the temperature, when the needle potential is 50 volts and the applied potential is 8.11 volts; this potential

is equivalent to 5 volts for the perpendicular specimen of a thickness of 0.294 mm. In this case the discontinuous

Fig. 21.

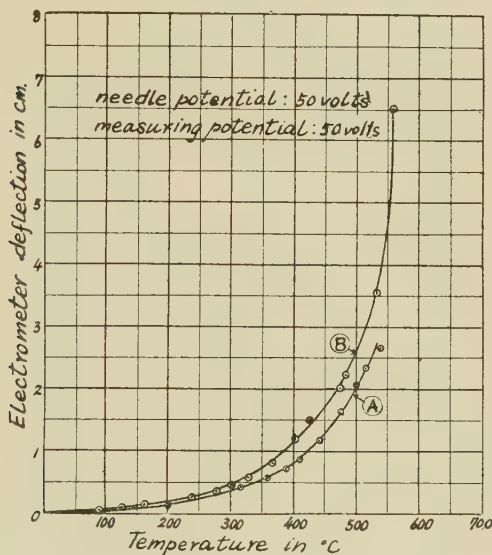
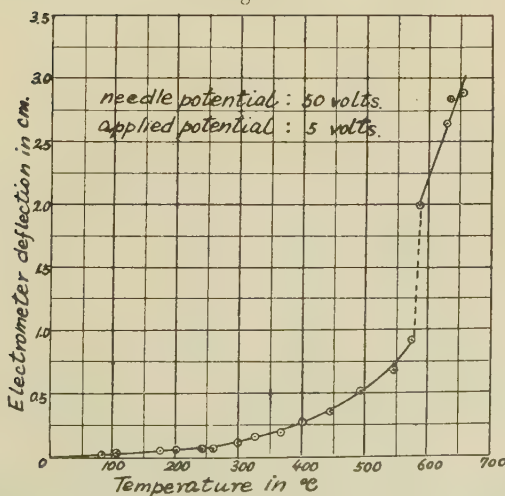


Fig. 22.



change at the temperature 573°C. is remarkably large compared with the case of fig. 22.

Fig. 23.

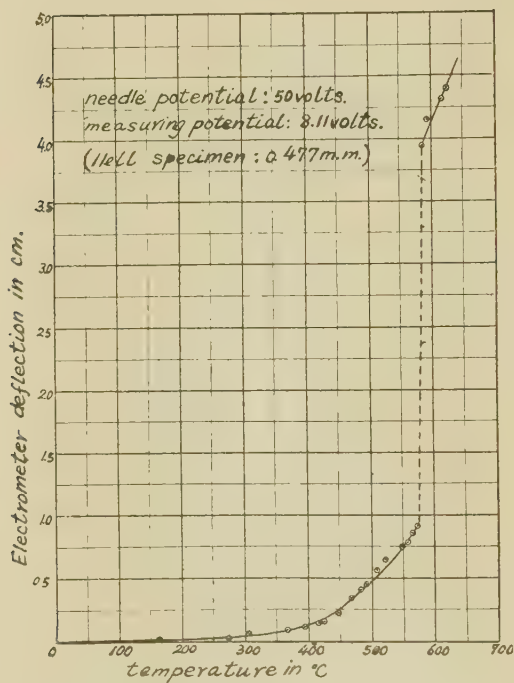


Fig. 24.

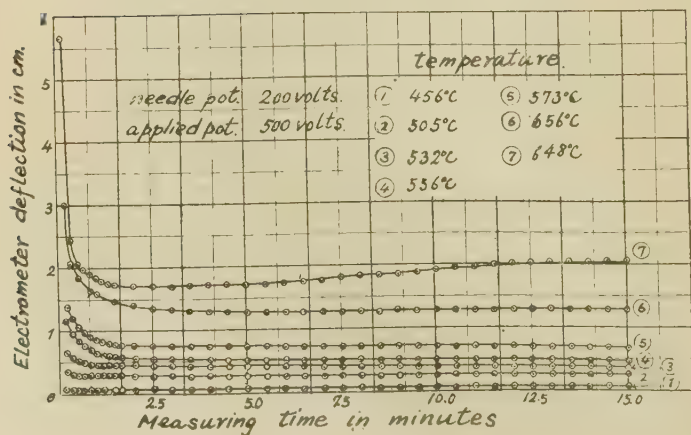


Fig. 25.

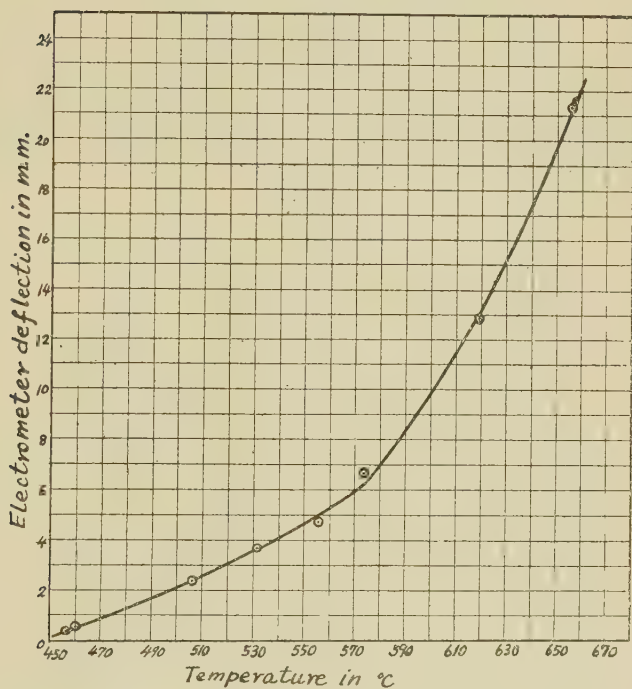
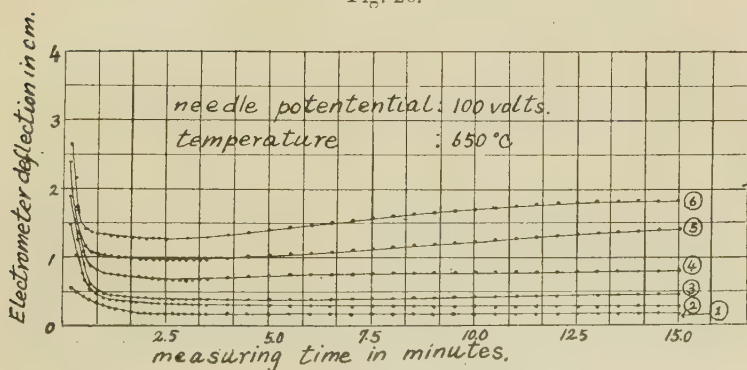


Fig. 26.

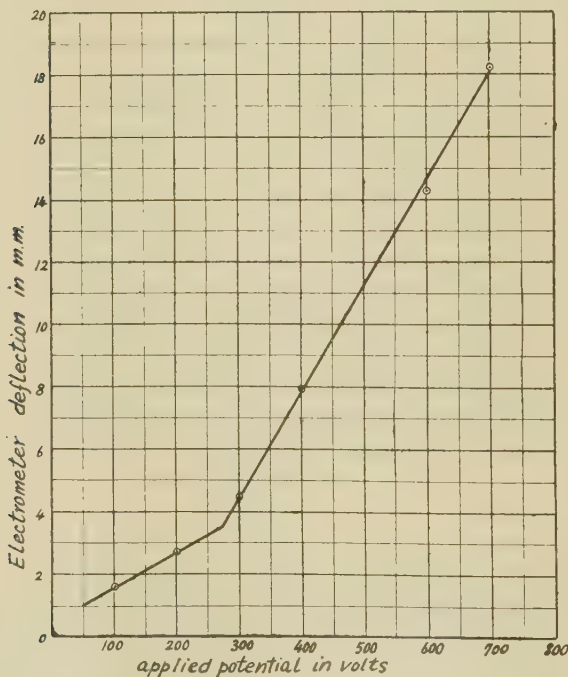


Measuring potential.

- | | |
|----------------|----------------|
| (1) 100 volts. | (4) 400 volts. |
| (2) 200 " | (5) 600 " |
| (3) 300 " | (6) 700 , |

Next we measured the conductivity by the potential fall method as in the case of the quartz perpendicular plate. The standard resistance used in this experiment is the same as in the former case. Fig. 24 shows the relation between the potential difference and the time for various temperatures, when the measuring potential is 500 volts and the needle potential is 200 volts; and fig. 25 shows the stationary value of the potential difference and the temperature. As is

Fig. 27.

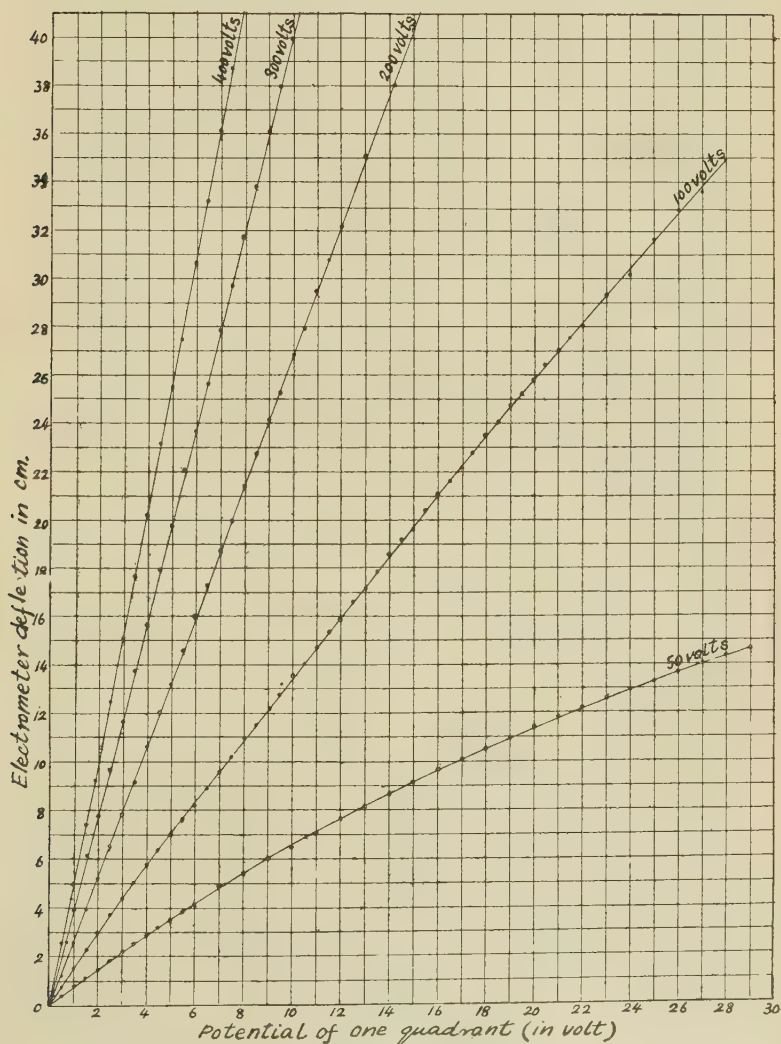


seen from the figure the variation of conductivity at the temperature 573°C . is quite of the same character as in the case of the quartz perpendicular plate, but in comparing with the case in fig. 23 there are remarkable differences. Nevertheless there exists no difference in both results, because in the former the initial value of conductivity is taken, but in the latter the stationary value.

Next we measured the conductivity for various applied potentials under a constant temperature. This result is shown in fig. 26. Fig. 27 was obtained from fig. 26, and shows the relation between the conductivity and the applied

potential. In this case the variation of conductivity also has a sharp point about its limit potential, as in the case of the quartz perpendicular plate.

Fig. 28.



Lastly, in order to compare absolute magnitudes on the curves, the relation between the potential of one quadrant and the electrometer deflexion at each needle potential is shown in fig. 28.

§ 4. *Concluding Remarks.*

1. The variation of electrical conductivity of quartz plates, one cut perpendicular the other parallel to their optical axes, with temperature for various measuring potentials, is carefully measured with various precautions, and we learn that the conductivity increases greatly with the temperature, and at the transformation point increases abruptly or discontinuously. It is a well-known fact that the refractive index, expansion coefficient, and other physical properties discontinuously change at the transformation point; in the present experiment it is found that the electrical conductivity shows also a similar behaviour at the transformation point.

2. As the result of our accurate measurement the conductivity increases gently up to 573°C. , then abruptly increases with the temperature above 573°C. , and the transformation point is markedly defined by the measurement of conductivity.

3. At high temperatures above the transformation point the electrical conductivity increases also with the measuring potential, and at the limiting potential the curve transforms to another form; thus the conductivity shows the anomalous change at the limiting potential.

4. The oxidation of the electrode-metal affects seriously the conductivity, *i.e.*, it makes a sharp variation at the transformation point smooth.

In conclusion, I am greatly indebted to Prof. H. Saegusa, under whose kind guidance the present experiment has been carried out.

September 1931.

LXXXVII. *Decay of Torsional Oscillation of an Iron Wire*
Effect of Variation of Period. By F. W. ALEXANDER,
B.Sc., Research Student, University College, Dundee.*

IT has been shown by Prof. Peddie (Phil. Mag. 1894) that the law of torsional oscillation of wires can be extremely accurately represented in many cases by the formula

$$y^n(x+a) = b,$$

where n , a , and b are constants for any one experiment, y the range of oscillation, and x the number of oscillations

* Communicated by Prof. W. Peddie, D.Sc., F.R.S.E.

which have taken place since torsion was first applied and the wire left to itself so that the oscillations gradually diminished. The present investigation was undertaken to find out the variation in the constant n when the period of oscillation, *i. e.*, the moment of inertia of the system, was varied, the length of the wire, the load on the wire, and the torsion applied being constant.

The apparatus used consisted of a stout wooden arm, rigidly fixed to the wall, holding the hollow brass socket of a torsion head, which consisted of a brass cone with a central pin-vice. The angle through which the head could be turned was limited by two projecting screws. The lower end of the wire was fixed symmetrically by a similar pin-vice to the oscillator, which consisted of a long horizontal hollow cylinder, with two solid lead cylinders of the same diameter, capable of sliding within it, fixed symmetrically at each side by two screws projecting through a long slot in the cylinder. The moment of inertia of the system could be varied by fixing the lead cylinders at any distance along the hollow cylinder. Attached to the cylinder with the pin-vice as centre was a thin metal circular ring, carrying a graduated scale. The maximum angular rotations about the wire were observed by means of a reading telescope.

In each experiment, with the lead cylinders clamped at a measured distance from the ends of the cylinder, torsion oscillations were started by turning the torsion head in and out five times between the limits fixed by the projecting screws, each turn being timed so as to amplify the swing. The system was then left to itself, and readings were taken through the telescope of the maximum amplitude of oscillation, first on one side and then on the other. By subtracting these two readings the amplitude of the complete oscillation could be found. A curve was then plotted with scale readings as ordinates and the number of swings as abscissæ; the latter have already been shown to be almost isochronous (*Phil. Mag.* 1894).

The method adopted for finding n was that given by Prof. Peddie (*Trans. R. S. E.* 1896). Using the formula given above we obtain

$$n \log y + \log (x+a) = \log b.$$

If $\log (x+a)$ be plotted against $\log y$, the points lie on a straight line which intersects the axis along which $\log y$ is measured at an angle whose tangent is n if the proper value of a is chosen. If a wrong value of a is used the points will not lie on a straight line. If a is too large the

curve is convex towards the origin and concave if a is too small.

The moment of inertia of the system was found by observing the periodic time of torsional oscillations of small amplitude. The two similar brass disks of known dimensions and mass were placed symmetrically on each side of the circular ring, and the periodic time was again found as before. From these two periodic times and the calculated moment of inertia of the disks about the wire the moment of inertia of the system was found.

An iron wire thinly coated with copper, 55.45 cm. in length between the two pin-vices, was used. A series of

TABLE I.

No.	Date.	Distance of Pb cylinders from ends.	Time of oscillation.	Moment of inertia.	Value of n .	Average value of n .
		cm.	sec.			
4.....	25/4/29	6	9.9	288,000	.647	.657
5.....	26/4/29	6	9.9		.668	
6.....	14/10/29	8	8.5	192,000	.629	.645
7.....	15/10/29	8	8.5		.650	
8.....	21/10/29	10.6	7.26	129,000	.621	.621
9.....	24/10/29	0	14.65	627,000	.671	.662
10.....	29/10/29	0	14.65		.654	
11.....	4/11/29	2.5	12.68	434,000	.623	.630
12.....	5/11/29	2.5	12.68		.647	
13.....	14/11/29	5	10.63	252,000	.633	.633
14.....	21/11/29	6.5	10		.592	.579
15.....	22/11/29	6.5	10		.602	
16.....	7/1/30	0	14.46		.602	.609
17.....	9/1/30	0	14.64	627,000	.617	
18.....	20/1/30	7	9.63	240,000	.546	.549
19.....	21/1/30	7	9.63		.553	

experiments spread over a considerable period of time was carried out, with as many different periods of oscillation as possible, care being taken to avoid torsional frequencies which were in too close agreement with the natural pendulum frequencies. This precaution was necessary because the energy of torsional oscillations was apt to be transformed into energy of pendular oscillations which destroyed the accuracy of the readings. Thus, the lead cylinders could only be placed at certain positions, found by trial and error, along the hollow cylinder. Table I. gives the results.

The value of the constant a varied between 30 and 50. As the maximum value of y was not large, only one value of n was found for each experiment.

It will first of all be noticed from Table I. that, if an experiment was repeated on the following day, the determination gave a higher value of n (*vide* nos. 4 and 5; 11 and 12, etc.), but if an interval of a week or more intervened the second gave a lower value of n (*vide* nos. 9 and 10). Nos. 9 and 10, carried out on 24th and 29th Oct., 1929, with the maximum moment of inertia, gave an average value of n equal to $\cdot 662$; but nos. 16 and 17 gave the value $\cdot 609$ for the same moment of inertia repeated on 7th and 9th January, 1930. From the above values of n no law between the period of oscillation or moment of inertia and n is apparent, any connexion being masked by the decrease of n with time. The physical condition of the

TABLE II.

No.	Date.	Distance of Pb cylinders from ends.	Time of oscillation.	Value of n .	Average value of n .
		cm.	sec.		
20	11/3/30	3.5	12.17	$\cdot 586$ }	$\cdot 577$
21	11/3/30	3.5	12.17	$\cdot 571$ }	
22	11/3/30	5	10.6	$\cdot 574$ }	$\cdot 573$
23	12/3/30	5	10.6	$\cdot 573$ }	
24	12/3/30	8	8.5	$\cdot 558$ }	$\cdot 561$
25	12/3/30	8	8.5	$\cdot 563$ }	
26	13/3/30	9	7.9	$\cdot 564$ }	$\cdot 564$
27	13/3/30	9	7.9	$\cdot 564$ }	
28	13/3/30	1.5	13.45	$\cdot 575$ }	$\cdot 575$
29	14/3/30	1.5	13.45	$\cdot 575$ }	
30	14/3/30	0	14.63	$\cdot 572$ }	$\cdot 572$
31	14/3/30	0	14.63	$\cdot 572$ }	

wire must have gradually altered as a result of the torsional strain to which it was subjected as time passed.

It was next decided to carry out another series of experiments, taking precautions to keep the wire in as nearly as possible the same physical state. The wire at the beginning of each day's work was fatigued slightly by being set in torsional oscillations which were allowed to decay. Then as many sets of readings as possible were taken in one day, the whole series being done in the consecutive days of one week. The plotting of the graphs and the calculations were carried out later. Table II. gives the results.

It is evident that, within the limits of experimental error, there is no variation in the value of n as the period of oscillation is altered by change of the moment of inertia, the period being varied between 7.9 sec. and 14.63 sec. The moment of inertia was not calculated for this series.

It has been shown (Phil. Mag. 1894) that large torsional oscillations of a wire cause the rupture of molecular groups in the wire, thus resulting in a loss of potential energy. But theory indicates that the value of n is increased if more groups break down under given conditions. Thence it would appear from the constancy of n that on the average the same number of molecular groups are broken up irrespective of the period of oscillation, *i. e.*, irrespective of the *speed* of deformation of the groups within the observed range.

In conclusion, I wish to thank Prof. W. Peddie and Dr. J. Forest for their guidance and helpful advice throughout this investigation.

LXXXVIII. *On certain Variations in the Optical Constants of Copper.* By H. LOWERY, Ph.D., F.Inst.P., Head of the Department of Pure and Applied Physics, and R. L. MOORE, M.Sc.Tech., Technological Research Scholar, College of Technology, University of Manchester*.

I. GENERAL DISCUSSION.

ON the initiation of a research aiming at the determination of relationships (if any) between the optical properties of a series of alloys and their crystalline constitution, it soon became evident, in view of the disparity between the results of previous observers, that a general survey of the methods, technique, and apparatus for the determination of the optical constants of metals was desirable.

Numerous workers (*e. g.*, Jamin⁽¹⁾, Quincke⁽²⁾, Drude⁽³⁾, Minor⁽⁴⁾, Meier⁽⁵⁾, Tool⁽⁶⁾, Tate⁽⁷⁾, and many others) have given data from which the refractive index, absorption, and reflexion coefficients of a metal may be deduced, and their methods of measurement have usually been based on either the transmission or reflexion of light. In illustration of the manner in which transmission methods have been used, reference may be made to the work of Kundt⁽⁸⁾, Pflüger⁽⁹⁾, Shea⁽¹⁰⁾, and du Bois and Rubens⁽¹¹⁾. These observers determined the refractive index of various metals by sending light through a very thin prism of the substance under investigation and measuring the deviation of the

* Communicated by the Authors.

beam. The technique of methods involving light transmission is extremely difficult, and the collected results show considerable variations among themselves. Drude⁽¹²⁾, after having made a very careful survey of the various methods of determining the optical constants of metals, came to the conclusion that *reflexion* methods are likely to prove more satisfactory than *transmission* methods for the study of the optical properties of metals, mainly on account of the ultra-refined technique and consequent practical difficulties incidental to the latter.

Leaving on one side the direct method, in which the reflecting power of a metal is measured by determining the proportion of the incident light reflected, using, say, a bolometer (*e.g.*, Hagen and Rubens⁽¹³⁾, and Coblentz⁽¹⁴⁾), reflexion methods usually consist in determining the form of the ellipse of vibration in a beam of light reflected from the polished surface of a metal. The standard method since the work of Drude is to reflect a beam of light polarized at an azimuth of 45° to the incident plane, and then to determine the phase change (Δ) between the reflected components in and perpendicular to the incident plane. To complete the data specifying the ellipse of vibration of the reflected light, the azimuth (ψ) of the reflected light after restoration to plane polarization is measured. Various devices have been used for the purpose; thus, Haughton⁽¹⁵⁾ and others have used Jamin's modification of Babinet's compensator. Drude⁽³⁾ has employed the Soleil modification of the same compensator, and his example has been followed by many more recent workers. Tool⁽⁶⁾ has evolved a very accurate half-shade analyzing system, and Minor⁽⁴⁾, Meier⁽⁵⁾, Weld⁽¹⁶⁾, and others have used a photographic method due to Voigt⁽¹⁷⁾.

These methods enable the two quantities Δ and ψ to be measured for any metal surface, from the values of which the three optical constants, viz., the refractive index (n), absorption coefficient (k), and reflexion coefficient (R) may be deduced.

The truth of the formulæ from which the optical constants are deduced is proved by a general rather than exact agreement between the results of polarimetric measurements and those of transmission methods. Tate⁽⁷⁾ has determined the reflecting coefficients of several metals by direct measurement and also polarimetrically, using the *same surface*, and has found an almost exact agreement between the two sets of measurements. Thus it seems reasonable to suppose that the values of n , k , and R , derived from polari-

metric measurements, are at least the correct values for the particular surface under consideration. Drude⁽¹²⁾ has also shown that the errors of measurement are least when observations are made at an angle of incidence for which Δ is approximately 90° ; but despite very elaborate precautions and a wealth of experimental skill the polarimetric observations of various observers show serious discrepancies. These variations have been noticed by most observers since the time of Biot, and Wheeler⁽¹⁸⁾ has illustrated the extent of the divergencies in a striking manner by graphing the results of several investigators for silver, copper, gold, nickel, and cobalt. It is generally recognized that it is almost impossible to reproduce a polished surface giving the same constants even by the most careful repetition of the conditions of polishing.

The results of Drude, Minor⁽⁴⁾, and Tool⁽⁶⁾ indicate that these variations are due to

- (a) impurities picked up in polishing;
- (b) an incomplete polish;
- (c) films produced on the test surface.

Drude investigated this latter effect theoretically⁽¹²⁾, and showed that a film of impurity tends to lower the value of Δ considerably and increase the value of 2ψ very slightly. From this it is concluded that the "standard" or "best" condition for a surface is such as to produce a maximum value of Δ for the particular angle of incidence used. Unfortunately the attainment of standard conditions is by no means as easily obtained as this. Wheeler⁽¹⁸⁾ says that "there seems to be at least, tacitly, a general assumption that since Drude's specification of the 'normal' condition for a reflecting surface, it only requires sufficient care in the preparation of the metallic mirror in order to get consistent results; and that consequently our knowledge of the optical constants is nearly as precise as that of the methods of measurement used. It is safe to say, however, that no one who has had much experience in this line of work has held such an opinion for long. In fact, experience teaches that it is well nigh impossible to exactly reproduce (even by the same observer) the same condition of the reflecting surface in different samples of the identical material, or even in the same sample at different times. Thus we cannot regard the experimental values of the optical constants as anything but rather rough approximations to the specific values characteristic of the substances."

With such acknowledged discrepancies it would seem desirable to investigate more closely the various factors which contribute to the general non-agreement of the various observations. It seems fairly certain that impregnation of the surface with impurity during the polishing processes is one of the root causes, but little work has been done with a view to arriving at a complete statement of all the factors affecting the results. Beilby⁽¹⁹⁾ has developed a theory of polish which seems to throw some light on the question. According to Beilby, polishing, as distinct from grinding, consists in the flowing of a film of metal until all the ridges and crevices produced by previous grinding are filled up. Evidence has been put forward to show that this film is amorphous. The work of G. P. Thomson⁽²⁰⁾ on electron diffraction by polished surfaces probably means that a polished surface is structureless to a depth of several atoms at least. Ignoring for a moment the question whether this layer is truly amorphous or not, it is well known that its chemical and mechanical properties are different from those of the unstrained crystalline metal. Any kind of cold work tends to increase the electrical resistance of a metal, and from these facts it appears that actual polishing may produce a surface which is not truly characteristic of the metal. It has, in fact, been suggested by Tool that this is the case. The work of Weld, Sieg, and others on small single metal crystals has shown quite definitely that the optical constants vary according to the direction of the axis of the crystal; so metallic reflexion is evidently dependent in some way on the arrangement of atoms as well as on any inherent properties of the atoms themselves. It remains to see whether this flowed metal is of sufficient depth to cause any appreciable change in the constants. Beilby⁽²¹⁾ was able by polishing and etching to prepare *transparent* films which he estimated to be between 10 and 20 $\mu\mu$ thick. Such films Beilby considered to be very thin, and he mentions that it is possible to vary the thickness of the film very considerably. His work suggests that polishing may produce either amorphous films or layers of distorted crystals with a minimum thickness of 10 to 20 $\mu\mu$.

If we now consider the depth of metal which is concerned in optical reflexion we can estimate the extent to which variations in the thickness of the film could possibly affect optical measurements. Hagen and Rubens⁽¹³⁾ have measured the amount of light reflected at normal incidence from gold films of varying thickness, and from their results it

is clear that the amount of light reflected reaches a maximum for red light when the thickness is about $80\ \mu\mu$. The absorption coefficient (k) for copper is about half that of gold, so we can safely say that the maximum depth of metal concerned in reflexion from copper is something between 150 and $200\ \mu\mu$. This, then, suggests the possibility of variations in the optical properties of the metal due to the formation of poor conducting layers of metal of thickness greater than 10 to $20\ \mu\mu$. As quite opaque layers can easily be produced, 10 to $20\ \mu\mu$ will probably be a low limit for the thickness of this layer.

As far as we are able to discover, Margenau⁽²²⁾ is the only observer who has made an attempt to vary the degree of polish in order to find any relationship between the amount of surface disturbance and the optical constants. He carried out direct measurements on the reflecting power of silver in the ultra-violet, and came to the conclusion that a light polish increased the reflecting power of a metal. This is presumably what would happen if a light polish left the greater portion of the metal concerned with reflexion in its normal high-conducting state, and a heavy polish produced a poorer conducting layer of much greater depth. Sieg⁽²³⁾ has made direct measurements of the reflecting power of small single crystals of selenium, and has obtained, as might be expected, different reflecting powers according as the long axis of the crystal was parallel or perpendicular to the electric vector of the incident beam. He points out that it might be expected that the results for a polished plate would be the mean of his extreme values. Comparing his results with those of Foersterling and Freedericksz⁽²⁴⁾ and Pfund⁽²⁵⁾ for polished cast plates, he shows that this is not so and that the values of the reflexion coefficient of these other observers are *below* the mean value—in fact, for part of the spectrum below the recognized minimum value. He suggests that this difference may be caused by the preferred orientation of crystals to a cast plate of selenium; but it is clear from Sieg's work that, using small crystals, the reflecting coefficient is at least not *increased* by polishing.

On the other hand, Drude's⁽²⁶⁾ result for a natural crystal of lead sulphide (probably a much larger crystal) shows that the reflexion coefficient is increased by polishing, and Tate⁽⁷⁾, on some work on electro-deposited *unpolished* gold and copper surfaces found absorption and reflexion coefficients considerably smaller, and the refractive index higher, than those of Tool.

With these apparently contradictory results it would seem desirable to estimate the extent and direction of changes in the optical constants with varying degrees of heaviness of polish. The first part of the present work is an attempt to discover some relationship between the state of strain of a surface and the optical constants using a polarimetric method, and the second part attempts to assess the rate at which atmospheric deterioration takes place. In both cases copper was the metal examined.

II. EXPERIMENTAL.

1. *The Method of Production of Varying Degrees of Polish.*

It has been pointed out that several causes for the variations of optical constants are already known and agreed upon by all observers, and that it is almost impossible to control the chief factors successfully. Thus, with several factors acting it may be in the same or opposite directions, a search for a further factor is attended by serious difficulties. The three main factors which it was desired to keep constant throughout all changes of degree of polish were

- (a) the purity of the surface ;
- (b) the completeness of the polish ;
- (c) the state of a film of foreign matter (if any)
between the metal surface and the air.

(a) According to Drude⁽¹²⁾ the test for a surface which is perfectly free from impurities is that Δ shall be a maximum. In this case the variations which the experimentally changed conditions may produce will possibly involve changes in Δ . This at once introduces an element of uncertainty which can only be eliminated by a reproduction of the results several times.

(b) Drude's⁽¹²⁾ criterion for perfection of polish is that the scratches in the surface should not give values of 2ψ for perpendicular and parallel positions differing by much more than about a degree, and that all the scratches should be in the same direction. Such conditions were again difficult to maintain, but a fairly large mirror was used, and the surface explored by a microscope until an area of about 6×6 mm. was found, which would satisfy these conditions. The rest of the mirror was then covered with black paper.

(c) The transition layer it was hoped would be very small, and an attempt was made to keep it constant by making the final processes of polishing always the same.

2. *The Production of the Test Surfaces.*

Two specimens of copper were used, obtained from quite different sources. Specimen A was prepared in a high-frequency induction furnace, in air, from electrolytic copper surrounded with plenty of carbon. The metal was only just allowed to melt and the furnace switched off. By this means a casting fairly free from blow-holes was produced.

Specimen B was a piece of cold-drawn high-conductivity copper rod obtained from the Broughton Copper Company, Manchester. It was entirely free from all blow-holes, and enabled very perfect mirrors to be produced. The impurity in both specimens was chiefly oxide in small inclusions at the crystal boundaries.

3. *The Unstrained Surface.*

The metal was planed up quite smooth on a shaping machine and then ground on a series of emery papers of increasing fineness placed on a glass plate. The final paper was a French 0000 paper. After this, since the processes had probably cold-worked the metal, it was annealed in the induction furnace under a pressure of 4 cm. of hydrogen. The surface was very clean, and it was only necessary to grind very lightly on the 0000 paper before polishing.

To produce a mirror with a thin cold-worked layer some polishing process must be used which produces a polish with as little surface flow as possible. Alumina polishing satisfies these conditions by its abrasive as well as polishing properties. Polishing was done on a wheel covered with "Selvyt" polishing cloth impregnated with alumina mixed in a fine paste. The specimen was held down quite lightly, and as soon as the scratches had been removed it was washed and dried. Great trouble was experienced in preventing the alumina from scratching the surface beyond repair, and only about one-third of the polishes were of value. The specimen was now etched lightly (until small etching pits appeared all over the surface) in order to remove the flowed layer, and repolished for a very short time as lightly as possible. The reason for this is that the amount of cold work involved in polishing a light etch is considerably less than that necessary to remove scratches from an emery polish. The mirror was now washed with soap and water, then with

alcohol and carbon tetrachloride solution, in order to remove all traces of grease.

4. Heavily Polished Specimens.

The initial grinding process was in this case the same. The metal was not annealed, but was burnished by rubbing gently with a polished steel tool, using a rocking motion, after the manner of Drude. After this the metal was taken on to a piece of cloth and polished lightly with alumina. It was not, of course, a vigorous polish, but the metal was allowed to come into contact with the alumina, so that variations attributed to any impregnation of the surface might be prevented. The specimen was then washed in the usual manner.

5. Apparatus and Calibration.

The main part of the apparatus consisted of a Fuess No. 1 A Spectrometer-Goniometer *, a very complete description of which is given in Tutton's 'Crystallography,' p. 453 (1911).

The instrument consists of two arms capable of rotation through 360° about a massive centre pillar, after the manner of an ordinary spectrometer. By means of microscopes with micrometer eye-pieces the positions of the arms could be read to $10''$. The collimating arm was fitted with an adjustable slit and a Glan-Thompson nicol mounted on a vertical circle reading to minutes. The other arm carried a similar nicol and an attachment for a Jamin modification of Babinet's compensator. This arm could be fitted with either a telescope eye-piece for adjusting the position of the arm or a microscope eye-piece focussing on the compensator.

6. Preliminary Adjustment.

Since it was necessary to set the polarizing nicol at an angle of 45° to the incident plane, the scales of both nicols were calibrated with respect to the axis of the instrument. A small piece of plane glass was mounted on a levelling table and adjusted so as to be parallel to the axis of the instrument by making observations of the slit image with the telescope eye-piece. The polarizing nicol was set so as to polarize light as nearly as possible in the incident plane, and readings of the analyzing nicol for extinction were made on

* This instrument was formerly in the possession of Professor A. E. H. Tutton, F.R.S.

either side of the circle. Since the plane of polarization was originally practically in the incident plane, the reflected light near the polarizing angle was almost completely polarized in this plane, and thus the setting of the analyzing nicol necessary to polarize light exactly perpendicular to the axis of the instrument could be found. The polarizing nicol circle was calibrated against the analyzer by crossing them several times. By this means the position of both circles with respect to the main axis could be found.

Using a monochromatic illuminator, the yellow mercury line (5770 \AA) was focussed on to the slit of the goniometer and the compensator calibrated by making settings on the -2π , 0, and $+2\pi$ fringes. Owing to a small amount of parallax, caused by imperfect alignment of the nicol, all settings of the compensator were made with the analyzing nicol in a fixed position. The calibration was repeated, and in all 440 settings were made, each of the two calibrations giving the same result to one part in 3000. The mirror was mounted in a black metal cup, which could be adjusted by means of a rocking table to be parallel to the axis of the instrument and also to be capable of rotation about a normal to its surface. Observations were confined to an angle of incidence which made Δ approximately 90° . Δ was calculated by measuring the shift of the fringes.

The angle 2ψ was determined by setting the analyzing nicol for maximum blackness of the fringes with the polarizing nicol in each of the four positions at 45° to the incident plane. Considerable trouble was experienced in making these readings, but the setting became more critical when the intermediate light was cut down by interposing a small slit near the compensator. This allowed a single fringe to be viewed against a dark background.

7. Check Experiments.

There is little risk of error in measurements of Δ , but 2ψ may be altered by internal reflexions at the various transparent surfaces. This latter error was shown to be small by making a series of settings of the analyzing nicol with the polarizing nicol at two positions 90° apart, the arms of the goniometer being in the end-on position. As a further check on the whole method the constants for mercury were measured for sodium light. This was done chiefly on account of the fact that the question of polish does not affect the mercury constants, and also because Wheeler⁽²⁷⁾ has made an exact study of the mercury constants for sodium

light. His method was followed in detail and need not be described here. Our results (the mean of two separate determinations) are compared with those of other observers in Table I., those for Wheeler being calculated from his values for Δ and 2ψ .

It is seen that our results are consistent with those of other observers, the variation, as optical constants of metals go, being small. From the results of these checks it was assumed that the experimental technique was quite trustworthy.

TABLE I.
Optical Constants of Mercury. λ 5893.

Observer.	n .	k .	R.
Drude	1.73	4.96	0.784
Meier	1.624	4.406	0.753
Wheeler	1.56	5.07	0.806
H. L. & R. L. M. ...	1.604	4.803	0.767

8. *The Optical Constants of Copper.*

Specimen A was used as a check in case B, which was the specimen upon which most of the experiments were conducted, should owing to any cause give unreliable results. Specimen A was given a light polish; specimen B was given two separate light polishes, a moderate burnish and a heavy burnish. The heaviness of the burnish was very roughly estimated by the amount of time taken to produce the surface, and checked by determining the time the metal had to be etched to produce a recognizable structure when examined under a microscope. By this means an attempt was made to gain some idea of the relative magnitude of the flowed layer. The results are shown in Table II.

It will be noticed that the value of Δ seems to bear no very clear relationship to the degree of cold working of the surface. This is probably because its value is very greatly affected by surface films and traces of impurity, so that if any change occurs it is completely masked by fluctuations due to these other causes. The changes in 2ψ due to films and impurity are much smaller, and it is in this constant (2ψ) that we find large changes in value. This latter

TABLE II.

Optical Constants of Copper at $\lambda = 5770 \text{ \AA}$.

Expt. no.	Treatment and notes.	Etching time.	Angle of incidence.	Δ .	2ψ .	$2\psi_1 - 2\psi_2$.	n .	k .	R.
<i>Specimen A.</i>									
1.....	Light polish.	—	70°	90° 24'	78° 37'	1° 3'	0.500	2.37	0.737
<i>Specimen B.</i>									
2.....	Light polish.	—	71° 30'	94° 56'	80° 26'	1° 59'	0.539	2.88	0.798
3.....	Ditto. (Slight scratches in direction perpen- dicular to main scratches.)	4 secs.	70°	89° 33'	80° 7'	0° 45'	0.480	2.29	0.746
4.....	Light burnish.	25 secs.	70°	94° 2'	79° 2'	0° 10'	0.556	2.53	0.701
5.....	Heavy burnish.	Several minutes.	70°	87° 57'	77° 14'	1° 32'	0.590	2.23	0.637

constant appears to possess a higher value (by a degree or so) after a light polish than after a heavy burnish. In the third result 2ψ was probably a little smaller owing to slight scratching perpendicular to the main direction. Increasing heaviness of polish seems to lower the value of 2ψ and cause definite variations in the values of n , k , and R .

Despite the erratic fluctuations in the value of Δ there are quite definite decreases in k and R with increasing heaviness of polish and an increase in the value of the refractive index.

The evidence seems to point to an increase in the reflecting power as the surface was subjected to less vigorous method of polishing, and this result agrees with that of Margenau for silver. The divergence from Drude's and Tate's suggestions may only be apparent. Their surfaces were produced in quite different ways, but the weight of evidence does seem to point to the fact that a variation in the amount of cold work may be one of the causes in producing the discrepancies between the results of various observers on the optical constants of metals. It would also seem desirable in view of this to determine the optical constants of a metal after polishing as lightly as possible (consistent with a perfect surface) and not burnishing, as has been done by many workers.

9. *The Variation of the Optical Constants due to Atmospheric Deterioration.*

It was noticed in carrying out some experiments that a comparatively short exposure to air reduced the value of Δ very considerably. Since in measuring the dispersion curve the mirror would have to be kept in the atmosphere for some considerable time, a series of observations extending over two days was taken, with a view to testing the reliability of dispersion measurements (Table III.). Observations of Δ were the mean of eighty readings. The number of readings for 2ψ were reduced to forty on account of the eye-strain involved. This accounts for the somewhat erratic changes in this constant. In calculating the final values of n and k a mean value is taken for 2ψ , as the changes, although suggesting a slight increase, were not sufficiently in excess of the experimental error. Curves are given showing the fall in the value of Δ , n , and k with time (figs. 1, 2, and 3). It will be seen that the refractive index changes rapidly, but k is much less affected by variations in

the value of Δ . On account of the sharp fall of Δ and its effect upon n it is suggested that in measuring the optical

Fig. 1.

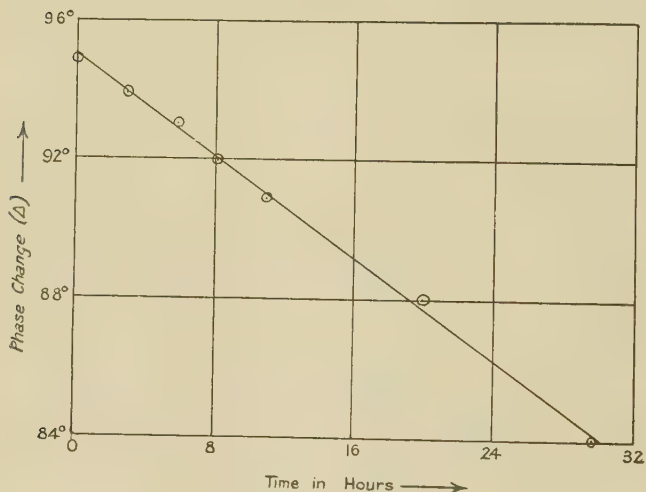
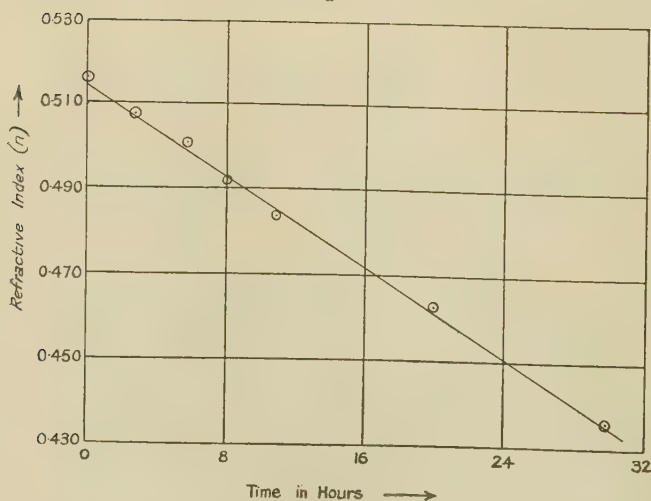


Fig. 2.



constants for several wave-lengths it would be advisable to measure all the values of Δ prior to measuring 2ψ .

Using this condition, measurements could be made with

the mirror remaining in air for several hours, and the points on the dispersion curve obtained consistent to within 1 or 2 per cent.

Fig. 3.

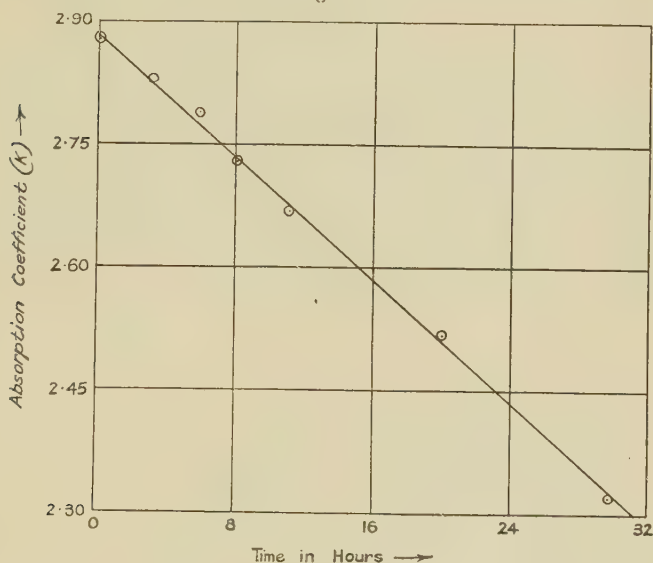


TABLE III.

Variation of the Optical Constants of Copper with Time.

Time.	Angle of incidence.	Δ .	2ψ .	Mean. 2ψ .	n .	k .
0 hrs.	$71^{\circ} 30'$	$94^{\circ} 56'$	$80^{\circ} 26'$	$80^{\circ} 50'$	0.516	2.88
2 hrs. 55 mins.	"	$93^{\circ} 54'$	$80^{\circ} 28'$		0.508	2.83
5 " 55 "	"	$93^{\circ} 8'$	$81^{\circ} 3'$		0.501	2.79
8 " 5 "	"	$91^{\circ} 59'$	$81^{\circ} 5'$		0.492	2.73
11 " 5 "	"	$90^{\circ} 57'$	$80^{\circ} 30'$		0.484	2.67
20 " 0 "	"	$88^{\circ} 0'$	$80^{\circ} 31'$		0.463	2.52
29 " 40 "	"	$84^{\circ} 2'$	$81^{\circ} 36'$		0.436	2.32

III. SUMMARY.

A brief survey of the methods of determining the optical constants of metals is given, and the discrepancies between

the results of various observers using polarimetric methods are discussed.

It is suggested that the Beilby theory of polish may offer an explanation of these divergencies, and to test this the optical constants of a piece of copper are measured for $\lambda = 5770 \text{ \AA}$, using varying degrees of surface strain. It seems evident from the results that a heavy polish increases the refractive index and decreases the absorption and reflexion coefficients. This agrees with Margenau's results for silver, but there seems to be an apparent deviation from Drude's and Tate's results. Some experiments were carried out on the rate of deterioration of a copper mirror, and from the results the most satisfactory condition for measurement of the dispersion curve (when the specimen must remain in the air for some time) is deduced.

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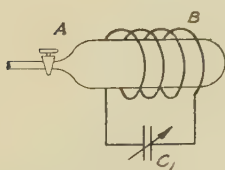
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- (2) Quincke, *Pogg. Ann.* Jubelband, p. 336 (1874).
- (3) Drude, *Wied. Ann.* xxxvi. p. 532 (1889); xxxix. p. 481 (1890); lxiv. p. 159 (1898).
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LXXXIX.—*Measurement of Current in Electrodeless Discharges by means of Frequency Variations**. By J. TYKOCINSKI-TYKOCINER, *Research Professor of Electrical Engineering, University of Illinois*†.

FOR quantitative investigations of electrodeless ring discharges the knowledge of current densities involved in this type of discharges is required. As no method is known which would serve this purpose an attempt was made to apply the principle of mutual reaction between two coupled circuits for the measurement of such currents.

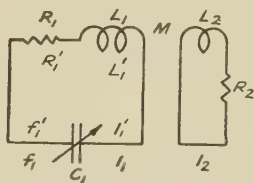
At first we will assume with J. J. Thomson that the phenomenon is due solely to electromagnetic induction. The rarefied gas in a tube or bulb A (fig. 1) inserted in the high-frequency field of a solenoid B becomes ionized, and may be regarded as carrying the secondary current of

Fig. 1.



Excitation circuit of discharge-tube.

Fig. 2.



Equivalent circuit of discharge-tube excitation.

a transformer whose primary current is oscillating in the windings of the excitation solenoid B. We should therefore expect that the ring discharge will react on the primary circuit, increasing its resistance and decreasing its inductance. Let the excitation be produced by continuous oscillations of constant amplitude (fig. 2), and let the current, resistance, inductance, and frequency of the primary circuit, when no discharge is taking place in the tube, be denoted by I_1 , R_1 , L_1 , and f_1 respectively. All of these magnitudes change when the discharge sets in, and acquire then the values I_1' , R_1' , L_1' , and f_1' , while the corresponding values for the discharge itself are I_2 , R_2 , L_2 , and $f_2 = f_1' = \omega_2/2\pi$.

* Paper read before the Am. Phys. Soc. Nov. 28, 1931. See Abstract in Phys. Rev. xxxvii. p. 100 (1930).

† Communicated by the Author.

The relation between these magnitudes is given by the expressions

$$\Delta R_1 = R_1' - R_1 = \left(\frac{\omega_2 M}{Z_2} \right)^2 R_2$$

$$-\Delta L_1 = L_1 - L_1' = \left(\frac{\omega_2 M}{Z_2} \right)^2 L_2,$$

$$(\omega_2/\omega_1)^2 = L_1/L_2,$$

and

$$Z_2^2 = L_2^2 \omega_2^2 + R_2^2,$$

which lead to the following formula for the increase of frequency caused by the reaction of the ring discharge upon the primary circuit, namely:—

$$\Delta f = \frac{2\pi^2 f_2^3 M^2}{(2\pi f_2 L_2)^2 + R_2^2} \frac{L_2}{L_1} \cdot \cdot \cdot \cdot (1)$$

Considering that

$$I_2 = I_1 \omega_2 M / z_2,$$

and that from (1)

$$(\omega_2 M / z_2)^2 = 2\Delta f L_1 / f_2 L_2,$$

we obtain for the secondary current

$$I_2 = I_1 \sqrt{2\Delta f L_1 / f_2 L_2}, \cdot \cdot \cdot \cdot (2)$$

an expression in which all parts with the exception of L_2 can be measured.

By inserting in (1) the coupling coefficient $K^2 = M^2 / L_1 L_2$ we obtain the relation

$$\Delta f = \frac{1}{2} \frac{f_2 K^2}{1 + \frac{R_2^2}{(2\pi f_2)^2 L_2^2}},$$

which may be used for the determination of the angle ϕ of phase difference between the electromotive force E_2 and the current I_2 ,

$$\cot \phi = R_2 / 2\pi f_2 L_2 = \left(\frac{K^2 f_2}{2\Delta f} - 1 \right)^{\frac{1}{2}} \cdot \cdot \cdot (3)$$

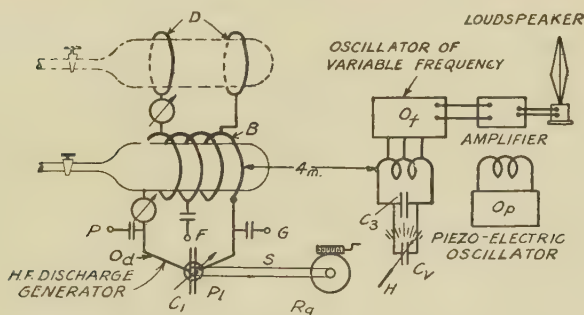
In deriving the above relations no attempt was made to consider the influence of the axial and radial distribution of the high-frequency currents in the tube. What was aimed at first of all was solely an approximate expression for checking the proposed method of current measurement, and also for verifying on the basis of J. J. Thomson's theory the picture usually drawn showing how currents are induced

within gases. A concise mathematical treatment of the problem, including the calculation of L_2 , will be given in a following paper by J. Kunz.

The experimental arrangement for a precise determination of frequency variations Δf , due to discharges induced in a rarefied gas, is shown in fig. 3.

The high-frequency discharge generator O_d consisted of two 75 Watt Ux-852 radiotron tubes, whose plate, filament, and grid were connected through blocking condensers to the solenoid B at the points P, F, and G respectively. This solenoid, together with the condenser C_1 , formed the primary excitation circuit. The solenoid B, 12.5 cm. in diameter and 12.5 cm. long, was made of twelve turns of nickel-plated copper strip 0.4 by 0.15 cm. thick. The variable air

Fig. 3.



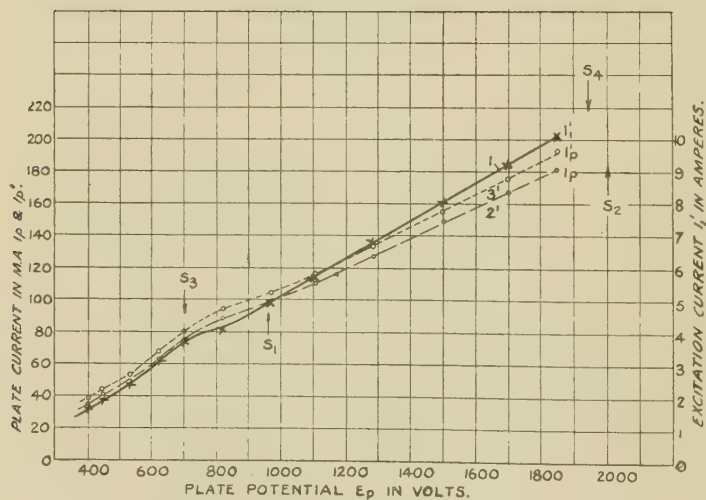
Arrangement of apparatus for the measurement of frequency variations.

condenser C_1 could be varied from 50 to 220 micro microfarads by means of a Vernier dial supplied with a pulley, P_1 . The latter was driven by means of a string S and reduction gear R_g , so that at a distance of 4 m. the frequency could be adjusted to correspond to that of a piezoelectric oscillator O_p , which served as a standard of fixed frequency f_1 . Coupled with the piezoelectric oscillator was another thermionic oscillator O_t , whose condenser C_3 could be varied over a large range of frequencies, while a Vernier condenser C_v in parallel connexion and supplied with a long handle H made possible finer adjustments for tuning O_t into resonance either with O_p or with O_d . Resonance was determined by the audible beat method. For this purpose the oscillator O_t was coupled with an amplifier and loud-speaker, or a head

telephone receiver was inserted directly in the plate circuit of O_t . All measurements were made at $f_1 = 3173$ k.c.

Two discharge-tubes made of pyrex glass and primed with electrolytically produced H_2 at a pressure of about 0.1 mm. Hg were used in the course of experiments. Both tubes had a length of 25 cm., while the diameters were 3 cm. and 6 cm. respectively. Any one of the tubes could be placed either inside the solenoid B, where the electromagnetic field predominated, or inside the rings D, where chiefly the electrostatic field was acting. By changing the two positions of the discharge-tube the influence of the character

Fig. 4.



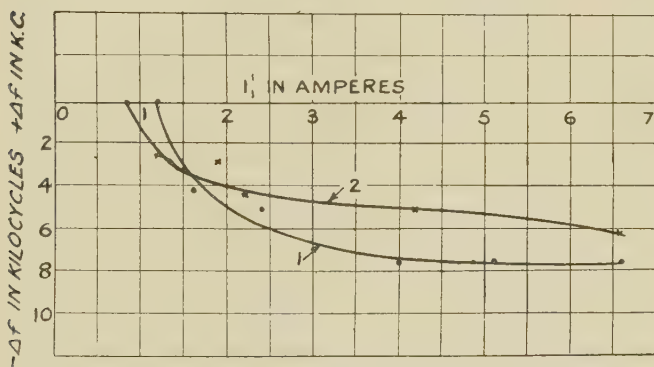
Relation between plate current, excitation current, and plate potential.

of the field could be studied. The intensity of the excitation could be varied by changing the plate potential at the 75 watt thermionic tubes. A storage battery of 2000 volts was used for this purpose.

In fig. 4 curve 1 represents the relation between the plate potential E_p and the oscillating current in the solenoid B before the discharge-tube is inserted. This current decreases slightly whenever the tube is inserted, which indicates that the discharge current reacts on the primary current. The reaction may be more distinctly traced by comparing the values of plate current I_p for the tube outside the solenoid (curve 2) with values I_p' when the tube is inserted (curve 3).

The procedure of the Δf measurement as function of the primary current is as follows:—The frequency of the oscillator O_d is set to correspond to the fundamental or to one of the harmonic frequencies of the piezoelectric oscillator O_p . Also the oscillator O_t is tuned into resonance with the same frequency. Then the discharge-tube is inserted in the solenoid B, and the plate potential adjusted below the point at which a discharge takes place. The frequency increase which is caused by the effect of the glass walls on the capacity of the circuit is next determined. When the condenser C_1 is re-set, so that again zero beats indicate resonance, the potential E_p is increased until a visible discharge sets in. The loud-speaker makes it audible by a sudden change in the pitch of the note. The capacity

Fig. 5.



Frequency variations with discharge-tube inside excitation solenoid.

of the Vernier condenser C_v of the oscillator O_t is then readjusted until the loud-speaker becomes silent, indicating again resonance. With the value ΔC_v thus determined the frequency variation Δf produced by the discharge is then computed from the relation $\Delta f = \Delta C_v f / 2c$.

By gradually increasing E_p the plate potential, the primary current I_1' , as also the secondary current I_2 induced in the discharge-tube, is increased. Consequently, if the theory holds, we should expect frequency increments which become larger with increasing values of E_p and I_1' .

Measurements were made with the tube 6 cm. in diameter as well as with the tube 3 cm. in diameter. In both cases the effect of an induced discharge was to decrease the frequency of the oscillator. In fig. 5 the values of Δf are

shown plotted against the primary current I_1' . Curve 1 represents values obtained with the small tube in a solenoid 7.5 cm. in diameter. Curve 2 shows values for the larger tube when inserted in a solenoid 12.5 cm. in diameter. Effects similar in character though smaller in magnitude were obtained when plane spirally wound coils were used instead of solenoids. Evidently the results of measurements do not support the theory, and are contrary to the calculations based on the simple assumption of electromagnetic induction.

The hypothesis may be made that in all cases investigated the electrostatic field inside the excitation coils causes a decrease of frequency. When the electrostatic field is strong as compared with the electromagnetic induction the trajectories of the electrons which move in the discharge-tube become complex and differ greatly from circular motions. Therefore the assumptions of the theory of electromagnetic induction cannot be applied for such general cases.

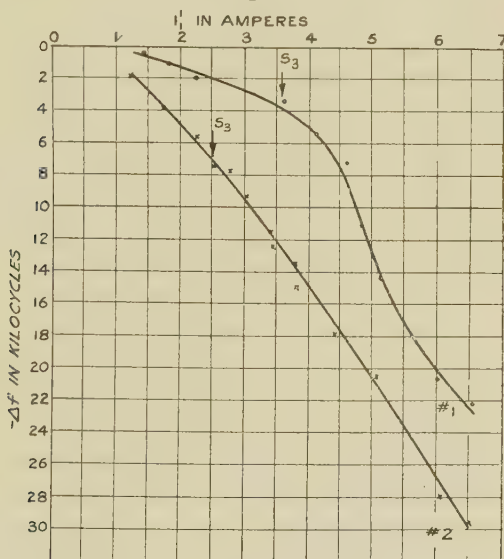
It was for the purpose of testing the above hypothesis that the solenoid in fig. 3 was supplied with two rings, D, supported by two metal strips 12 cm. long. The experimental procedure, as described, was repeated for the discharge-tube placed between the rings D, which served as outside electrodes for a purely electrostatic excitation. The curve in fig. 6 represents the value of frequency variation as function of primary current obtained for the same two tubes subjected to such an excitation. Again, the variations were found to be decrements of frequency, but their absolute values increasing at a much larger rate than in the case (fig. 5) when the tube was placed inside the solenoid B. In order to account for such a striking increase of the rate of change of Δf we must consider that in the portion of the tube inside the solenoid the gas is subjected to both the electrostatic and to the electromagnetic fields, which force the electrons to move in complex spirals. If, however, the tube is placed between the rings D the electrostatic field forces the electrons to move predominantly in lines parallel to the axis of the tube.

It appears as if two effects would combine inside the excitation solenoid, of which one produces a decrease the other an increase of frequency; indeed, it is possible to separate the frequency variations due to circular currents from those variations which are caused by the combined action of axial and circular currents in the tube. For this purpose it is necessary to establish between the rings D an electrostatic field which will be equal in intensity and distribution to that which exists inside the solenoid.

It may be assumed, on the basis of observations made in the course of many experiments, that it is always an axial field which by preliminary ionization causes the discharge to set in. If the magnitude of the potential which starts the discharge in each of the positions of the tube be made equal the electrostatic field in both cases may be regarded as similar.

In carrying out this method of separation the ring electrodes D were chosen of such a diameter (7.5 cm.) that the solenoid current I^1 , necessary for the discharge to just set

Fig. 6.



Frequency variations with discharge-tube outside excitation solenoid.

in, becomes of equal intensity for both positions of the tube. By slightly displacing the rings along the axis small differences in the starting primary current I may be eliminated. Slight discrepancies of the electrostatic field distribution inside the solenoid, as compared with the field between two rings separated by a distance slightly greater than the length of the solenoid, may be disregarded.

Under these conditions the electrostatic field which produces axial currents in the tube remains approximately the same for both positions. It does not matter for the magnitude of the axial current component whether the tube

is placed between the rings D or inside the solenoid B. In the latter case, however, a circular current component is produced additionally which influences the frequency variations. By subtracting the frequency variations in case of axial currents from those variations which are caused by the combined action of electrostatic and electromagnetic fields the two components can be separated and their relative values determined.

The result of such measurements on the larger tube (diameter 6 cm.) is represented by the curves *a*, *b*, and *c* in fig. 7. Hydrogen electrolytically produced from $\text{Ba}(\text{OH})_2$ at a pressure of 0.1 mm. Hg was subjected first to the combined action of electromagnetic and electrostatic fields inside the solenoid B. Curve *a* shows the decrement of frequency (Δf_a) as function of primary current I_1 . Then the same tube was inserted between the rings D. Curve *b* (Δf_b) shows the values of frequency decrements due solely to axial currents produced by the electrostatic field. The difference $\Delta f_c = -\Delta f_a - (-\Delta f_b)$, as shown by the curve *c*, gives the increase of frequency due to circular currents induced by the high-frequency electromagnetic field of the solenoid.

For the curve *c* an expression was empirically obtained

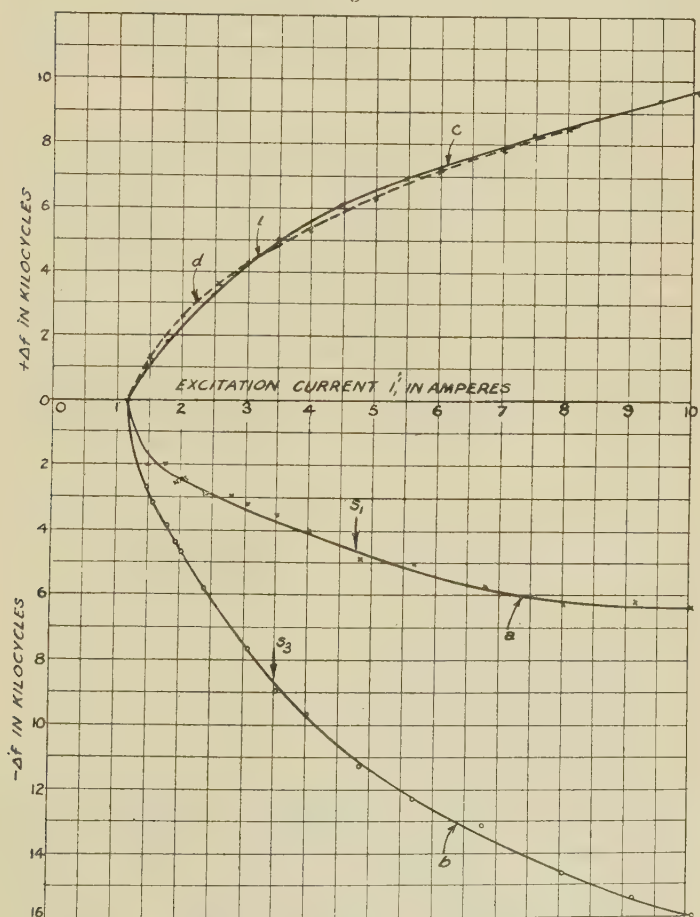
$$\Delta f = k \sqrt{I_1' - i}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $k=3320$ and $i=1.4$ amperes, *i. e.*, the value of the excitation current at which a discharge sets in. Both k and i are functions of frequency and pressure.

The maximum deviation of the experimental results (full line curve *c*) from the above expression (dotted curve *d*) is 9 per cent. In this connexion the character of the discharge as evidenced by its appearance may be significant. The discharge sets in at an excitation current of 1.4 ampere as a pinkish-grey uniform glow, whose luminosity increases with the intensity of excitation. At a certain critical value of the current I_1' striations become visible. First they are of an unstable character, indefinite in number, and moving to and fro along the axis of the tube. By a slight increase of the current the striæ become stable. They appear then in pairs six to eight in number. Increasing the excitation current decreases the number of pairs of striæ. They continue to disappear one by one until at another critical value of the excitation current a uniform glow is again established, but of a more intense luminosity than that of the first stage. The striæ appear when the tube is excited by the combined action of electromagnetic and electrostatic induction inside

the solenoid. They also appear when the tube is exposed to the action of electrostatic field alone between the external rings D (fig. 3). However, in case of the complex form of excitation the striæ start and disappear at larger currents than in the case of the simple electrostatic excitation.

Fig. 7.



Separation of frequency variations due to circular currents.

The values of the excitation current at which stable striæ appear are marked in figs. 4, 6, and 7 by arrows S_1 , while S_2 in fig. 4 indicates the current at which the latest pair of striæ disappears. Thus S_1 to S_2 includes the range of

current values connected with formation of stable striæ when the tube is placed inside the solenoid. Similarly S_3 to S_4 includes the range of stable striæ formation for the tube placed between external ring electrodes.

The deviations of the curve d from c may be correlated with the appearance of striæ at currents different in magnitude for each of the two types of excitation; indeed, the deviation is maximum at $I_1' = 4.75$ amperes near the region marked S_1 , and diminishes towards increasing as well as towards decreasing values of I_1' . The intersection of the curves c and d at e coincides with the zone S_3 .

Further measurements have shown that the decrease of frequency due to the axial current component increases with the length of the solenoid and with the frequency of the oscillator which excites the discharge tube. An explanation may be offered on the basis of our hypothesis. With increased frequency of the oscillator the distribution of current and potential in a coil approaches that of a standing wave. Potential antinodes are being built up at both ends of the solenoid. The larger its length the more these potential antinodes concentrate at the ends of the solenoid. The end windings act then like ring electrodes which project electrostatic lines of force through the cross-section of the solenoid.

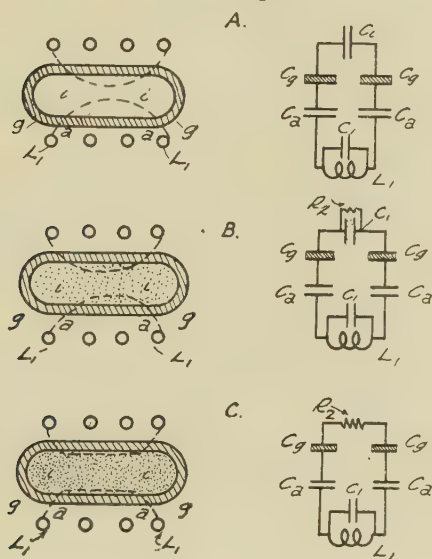
The question may be asked, how does the electrostatic field produce an increase of frequency? Fig. 8 serves to illustrate the influence electrostatic fields exert on the frequency of the oscillator.

Electrostatic lines of force in passing from one end of the coil L_1 to the other end penetrate three media, of which each possesses a different dielectric constant. The lines pass first through air, then through glass, then through ionized gas, and at last again through glass and air. A corresponding equivalent circuit A may be represented by a branch of condensers connected parallel to the condenser C_1 of the excitation circuit. This branch consists of a pair of condensers C_g , which represent the effect of glass walls of the tube, a pair of air condensers C_a , and a single condenser C_i to indicate the rôle of the ionized gas inside the discharge-tube.

The effect of the latter is composed of three parts; one is comparatively small and consists in the change of capacitance produced by the dielectric constant as function of ionization. The other part consists in the increased conductivity, which is equivalent to the decrease of a resistance R_2 shunted across the condenser C_i , as shown in the equivalent

circuit B. The third part of the effect is due to the redistribution of the electrostatic field with increased conductivity of the ionized gas. With increasing current inside the tube more of the electric flux concentrates in the glass, so that the capacity C_g increases. Gradually the gas conductivity becomes so large that the condenser C_i may be regarded as shorted and the equivalent circuit changed from B to C. The resultant capacitance of the branch C_a, C_g, R_2, C_g, C_a

Fig. 8.



*Equivalent Circuit of Discharge Tube
in the e.s. Field of a Coil.*

thus increases and causes a corresponding decrease of the frequency of the oscillating circuit.

The results of the investigation may be summarized as follows:—

1. By measuring frequency variations evidence was obtained for the existence of two current components, axial and circular, in discharges produced by high-frequency excitation of H_2 in a tube inserted inside a solenoid.

2. The axial component produces a decrease of frequency which may be determined by inserting the discharge-tube

within two external electrodes connected to the terminals of the solenoid.

3. The circular component produces an increase of frequency. The effect of circular currents can be separated from the effect of axial currents.

4. The intensity of the circular current and the phase difference between this current and the induced electromotive force can be determined from the magnitudes of frequency variations.

5. As a consequence of the experiments it became evident that the theory of induction as given by J. J. Thomson may be applied only to special cases when the induced currents are large enough to produce a conductivity in the gas sufficient to shield the electrostatic field of the excitation coil.

Following our investigation Prof. Chas. T. Knipp checked our results by qualitative experiments in which luminous effects due to the two current components were made visible by a method of shadows*. Not only do his experiments confirm the existence of circular and axial currents, but they visualize also the effect of simultaneous action of both currents in accordance with our theory.

In the following paper Prof. J. Kunz gives a theoretical investigation which led to more general expressions for the current in the discharge tube as function of the primary excitation current and of frequency variations. References of other authors who investigated electrostatic effects in electrodeless discharges are included in that paper.

Acknowledgment is due to the students (M. McGregor, R. L. Gougler, W. A. Laning, Jr., and D. E. Chapman) who at different times assisted in the experiments and measurements.

XC. Theory of Electromagnetic and Electrostatic Induction in Electrodeless Discharges. By JAKOB KUNZ, Professor in the University of Illinois†.

SIR J. J. THOMSON‡ has given a theory of the electrodeless ring discharge based on the assumption that the luminous current is caused by the electric force

* Chas. T. Knipp, *Phys. Rev.* xxxvii. p. 757 (1931).

† Communicated by the Author.

‡ *Phil. Mag.* iv. p. 1128 (1927).

due to electromagnetic induction. The formulæ of J. J. Thomson enable us to calculate the circular current in the discharge, a current, that appeared to be the only one produced in an experiment, which has not yet been measured. We therefore intended at first to measure this current by some indirect method. We shall see that the current can be determined by measuring the e.m.f. around the circuit and the change of the frequency in the primary circuit. Accepting the notations of Sir J. J. Thomson, we find the magnetic force in the discharge-tube given by

$$H = H_0 \sin \omega t \frac{J_0(jnr)}{J_0(jnR)},$$

the components of the current densities by

$$4\pi i_y = \frac{\partial H}{\partial z} = H_0 \frac{\sin \omega t}{J_0(jnR)} \frac{\partial J_0(jnr)}{\partial r} \cdot \frac{z}{r},$$

$$4\pi i_z = -\frac{\partial H}{\partial y} = -H_0 \frac{\sin \omega t}{J_0(jnR)} \frac{\partial J_0(jnr)}{\partial r} \cdot \frac{y}{r},$$

and the resultant density by

$$4\pi i = H_0 \frac{\sin \omega t}{J_0(jnR)} \frac{\partial J_0(jnr)}{\partial r} \cdot \frac{jn}{jn}.$$

The total induced current i_t per unit length measured along the axis is given by

$$4\pi i_t = 4\pi \int_0^R i dr = \frac{H_0 \sin \omega t}{J_0(jnR)} \int_0^R \frac{\partial J_0(jnr)}{\partial r} \cdot dr,$$

$$i_t = \frac{H_0}{4\pi} \sin \omega t \frac{J_0(jnR) - J_0(jn0)}{J_0(jnR)} = I \sin \omega t.$$

The magnetic flux ϕ through the tube is given by $\phi = \int H 2\pi r dr$, and the e.m.f. around a circle of radius r by

$$\begin{aligned} \text{E.m.f.}_r &= \int_0^r 2\pi r dr \frac{\partial H}{\partial t} \\ &= 2\pi\omega \frac{H_0 \cos \omega t}{J_0(jnR)} \int_0^r J_0(jnr) r dr \\ &= 2\pi\omega H_0 \cos \omega t \frac{r J_1(jnr)}{jn J_0(jnR)}. \end{aligned}$$

The e.m.f. around the circle of radius R is equal to

$$\begin{aligned} \text{E.m.f.}_R &= 2\pi\omega H_0 \cos \omega t \frac{R}{n} \text{ (when } Rn \text{ is large)} \\ &= E_a \cos \omega t. \end{aligned}$$

This e.m.f._R can be measured directly, its amplitude

$$E_a = 2\pi\omega H_0 \frac{R}{n}.$$

If we measure E_a , ω , H_0 , and R we can determine n . Knowing n , we can determine σ by means of $n^2 = j4\pi\omega\sigma$ or

$$n = \sqrt{j} \sqrt{4\pi\omega\sigma},$$

$$n = \frac{1+j}{\sqrt{2}} \sqrt{4\pi\omega\sigma},$$

and the real part of it is

$$n_{\text{real}} = \sqrt{2\pi\omega\sigma},$$

while ϕ_R is equal to

$$\phi_R = 2\pi H_0 \sin \omega t \int_0^R \frac{J_0(jnr)dr}{J_0(jnR)}$$

$$= 2\pi H_0 \frac{\sin \omega t}{J_0(jnR)} R J_1(jnR) = \phi_1 + \phi_2, \quad (1)$$

equal to the resultant flux through the tube. ϕ_1 is the flux due to the primary current i_1 through the coil, ϕ_2 due to the induced current $i_2 = i_t \cdot l$ in the discharge.

$$\phi_1 = H_0 \pi R^2 \sin \omega t,$$

$$\phi_2 = \pi R^2 H_0 \sin \omega t \left\{ \frac{2}{R} \frac{J_0(jnR)}{J_0(jnR)jn} - 1 \right\} = L_2 i_t l. \quad (2)$$

L_2 is the coefficient of self-induction of the current $i_2 = i_t \cdot l$. Substituting i_t of a previous equation in equation (2), we find

$$L_2 = 4\pi^2 R^2 \frac{2}{J_0(jnR) - J_0(jno)} \frac{J_1(jnR)}{jn} - J_0(jnR);$$

as we know σ and n , we can evaluate the Bessel functions and calculate L_2 . We consider the oscillating circuit of fig. 1. The potential difference between the condenser plates shall be p , the primary current i_1 , the number of turns per unit length N , the coefficient of self-induction of the system L , the resistance R_e and the coefficient of mutual inductance between the primary and the secondary currents M . The generalized Ohm's law gives the equation

$$p = i_1 R_e + L \frac{di_1}{dt} + M \frac{di_2}{dt} \quad \dots \quad (3)$$

Moreover,

$$H = H_0 \sin \omega t = 4\pi N J_1 \sin \omega t.$$

$$i_1 = I_1 \sin \omega t = -\frac{dq}{dt} = -C \frac{dp}{dt};$$

hence

$$\begin{aligned} \frac{H_0}{4\pi} &= N I_1, \quad i_2 = i \quad l \\ &= l N I_1 \sin \omega t \frac{J_0(jnR) - J_0(jno)}{J_0(jnR)}; \end{aligned}$$

As i_2 is opposite to i_1 , we write

$$i_2 = -l N i_1 \frac{J_0(jnR) - J_0(jno)}{J_0(jnR)}.$$

With these substitutions we obtain from (3)

$$\frac{d^2 p}{dt^2} \left[LC - MNC \frac{J_0(jnR) - J_0(jno)}{J_0(jnR)} \right] + R_e C \frac{dp}{dt} + p = 0.$$

Neglecting the second term, we may write the solution in the form $p = P \cdot e^{j\omega_2 t}$, where ω_2^2 is equal to

$$\omega_2^2 = \frac{1}{LC - MNC \left[\frac{J_0(jnR) - J_0(jno)}{J_0(jnR)} \right]},$$

while

$$\omega_1^2 = \frac{1}{LC};$$

hence

$$\omega_2^2 = \omega_1^2 \cdot \frac{1}{1 - \frac{MN}{L} \left[\frac{J_0(jnR) - J_0(jno)}{J_0(jnR)} \right]}, \quad (4)$$

or approximately

$$\omega_2^2 = \omega_1^2 \left[1 + \frac{MN}{L} \left(\frac{J_0(jnR) - J_0(jno)}{J_0(jnR)} \right) \right]. \quad (5)$$

Neglecting the resistance again, we obtain from (3)

$$p = L \frac{di_1}{dt} + M \frac{di_2}{dt},$$

or after the given substitutions

$$p = LI_1 \omega \cos \omega t - \omega M / N I_1 \cos \omega t f(\omega, \sigma, R),$$

where
$$f(\omega, \sigma, R) = \frac{J_0(j\mu R) - J_0(j\nu R)}{J_0(j\mu R)},$$

a function which approaches 1 with increasing frequencies.

$$p = I_1 \omega \cos \omega t [L - MNf(\omega, \sigma, R)] = P \cos \omega t;$$

hence the amplitude P of the potential difference is equal to

$$P = I_1 \omega [L - MNf(\omega, \sigma, R)]. \quad (6)$$

If we measure I_1 , the current, then we know P by this equation. From (5) we obtain

$$\frac{\omega_2^2}{\omega_1^2} - 1 = \frac{(\omega_2 - \omega_1)(\omega_2 + \omega_1)}{\omega_1^2} = \frac{\Delta\omega \cdot 2}{\omega_1} = \frac{MN}{L} f(\omega, \sigma, R), \quad (7)$$

or ω_1 , which is more nearly equal to ω , is given by

$$\omega_1 \simeq \omega \simeq \frac{2\Delta\omega L}{MNf(\omega, \sigma, R)}.$$

Substituting this in (6), we obtain

$$P = I_1 \frac{2\Delta\omega L}{MNf(\omega, \sigma, R)} (L - MNf(\omega, \sigma, R)),$$

or

$$P = I_1 2\Delta\omega L \left[\frac{L}{MNf(\omega, \sigma, R)} - 1 \right] \quad (8)$$

This formula shows how $\Delta\omega$ changes with P , the amplitude of the potential difference. Moreover, we obtain from (7)

$$\Delta\omega = \frac{\omega_1}{2} \frac{MN}{L} f(\omega, \sigma, R), \quad (9)$$

from which we can calculate $f(\omega, \sigma, R)$ and σ , the conductivity itself; but the specific resistance is $= \frac{1}{\sigma}$, and the total (ring) resistance of the tube is equal to

$$R_e = \frac{2\pi R/2}{\sigma R l} = \frac{\pi}{\sigma l},$$

or the resistance per unit length $= R_{e1} = \frac{\pi}{\sigma}$. The total induced ring current

$$i_2 = i_{\mu l} = l N I_1 \sin \omega t f(\omega, \sigma, R),$$

or by means of (9)

$$i_2 = l N I_1 \sin \omega t \frac{2\Delta\omega L}{\omega_1 M N} = I_2 \sin \omega t, \quad (10)$$

where the amplitude I_2 is equal to

$$I_2 = I_1 \frac{2\Delta\omega L}{\omega_1 M} \dots \dots \dots (11)$$

In this way the ring current can be determined. By (5) the frequency of the primary circuit ω_2 , when ionization is produced in the tube, is larger than ω_1 , the frequency without luminous discharge. The same statement holds when the luminous ring in the gas is replaced by a wire or by a ring filled with an electrolyte. This result is intuitively evident, because the effective self-induction L of the primary is decreased when a current in a neighbouring circuit is induced in a direction opposite to the primary current.

In order to estimate the order of magnitude of $\Delta\omega$ given by (9) we calculate at first $f(\omega, \sigma, R)$, assuming the following values :

$$R_e = \frac{1.6}{\sigma} = 500 \cdot 10^9, = 3 \cdot 10^{-12} \text{ e.m. unit.}$$

$$f(\sigma, \omega, R) = \frac{J_0(jnR) - J_0(jn\sigma)}{J_0(jnR)},$$

$$J_0(jx) = 1 + \frac{\left(\frac{1}{2}x\right)^2}{1!^2} + \frac{\left(\frac{1}{2}x\right)^4}{2!^2} + \dots,$$

$$J_0(jnR) = 1 + \frac{1}{4}n^2R^2 + \frac{1}{2^6}n^4R^4 + \dots,$$

$$J_0(jn\sigma) = 1,$$

$$f(\omega, \sigma, R) = \frac{\frac{1}{4}n^2R^2 + \frac{1}{2^6}n^4R^4 + \dots}{1 + \frac{1}{4}n^2R^2 + \dots},$$

but

$$n^2 = j4\pi\omega\sigma;$$

hence

$$f(\omega, \sigma, R) = \frac{\frac{1}{4}4\pi\omega\sigma jR^2}{1 + \frac{1}{4}4\pi\omega\sigma jR^2}.$$

Assuming

$$\omega = 2 \cdot 10^7, \quad \sigma = 3.2 \cdot 10^{-12}, \quad R = 2,$$

the real part of $f(\omega, \sigma, R)$ becomes approximately $= 8 \cdot 10^{-4}$.

$$\Delta\omega = \frac{\omega_1}{2} \frac{MN}{L} f(\sigma, \omega, R).$$

As an example we assume

$$\omega_1 = 2 \cdot 10^7, \quad M = 2 \cdot 10^{-6} \text{ henry}, \quad L = 2 \cdot 10^{-5} \text{ henry},$$

$$N = 4, \quad f(\omega, \sigma, R) = 8 \cdot 10^{-4},$$

and find

$$\Delta\omega = 3 \cdot 2 \cdot 10^8.$$

In the preceding paper, J. T. Tykociner found that in all cases ω_2 was smaller than ω_1 . This result shows that our theory can only explain a part of the phenomenon, and that the greater part in the change of ω is due to another cause.

It has been suggested by O. Lehmann*, E. Lecher, A. Steiner†, and more recently by J. S. Townsend‡, that the electrostatic forces between the ends of the coil are stronger than the electric forces due to electromagnetic induction, and K. A. MacKinnon§ came to the conclusion that the glow discharge in the electrodeless tube is largely electrostatic, while the ring discharge, investigated by Sir J. J. Thomson is of electromagnetic origin. In addition, C. J. Brasefield|| has shown that the luminous discharge ceased when the glass tube was shielded by a set of wires parallel to the axis of the tube joined at each end by a ring with a small gap in it. If in this way the tube is shielded and the current in the primary coil is strong enough, it should be possible to obtain a pure electromagnetic ring discharge.

We proceed to find out how the ionization due to electrostatic forces leads to a decrease of the frequency. In fig. 1 the potential difference $p_1 - p_2$ is equal to

$$p_1 - p_2 = \frac{d}{dt} (H \pi R^2 N l) = \pi R^2 N l \frac{dH}{dt},$$

and the axial electric force

$$E_{es} = \frac{p_1 - p_2}{l} = \pi R^2 N \frac{dH}{dt},$$

if we assume a uniform magnetic field H in the coil. In a

* Wied. Ann. xlvii. p. 427 (1892).

† Wiener Ber. cxiii. a, p. 403 (1904).

‡ Phil. Mag. v. p. 178 (1928).

§ Phil. Mag. viii. p. 605 (1929).

|| Phys. Rev. xxxiv. p. 1627 (1929).

conducting circle of radius R_1 in the tube, the e.m.f. induced is equal to $\pi R_1^2 \frac{dH}{dt}$, and the circular electric force

$$E_m = \frac{\pi R_1^2}{2\pi R_1} \frac{dH}{dt} = \frac{R_1}{2} \frac{dH}{dt};$$

hence

$$\frac{E_{es}}{E_m} = 2\pi N \frac{R^2}{R_1}.$$

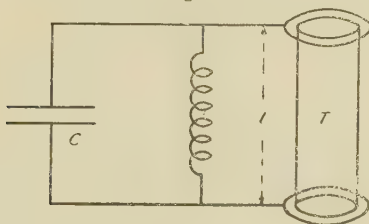
In the tube T with external electrodes we consider electrons moving under the influence of an alternating electric field,

$$E = E_0 \cos(\omega t + \phi).$$

The differential equation of motion of the electrons is as follows:

$$m \frac{d^2 x}{dt^2} = e E_0 \cos(\omega t + \phi),$$

Fig. 1.



and the conductivity is equal to

$$\sigma = \frac{e^2}{c} \frac{l}{m} \frac{n N_e}{\omega^2 + n^2} \text{ e.m.u.,}$$

where N_e is the number of ions per unit volume, n the number of collisions of one ion per unit time, ω the frequency per 2π seconds, e the charge, and m the mass of the ion. If we assume uniform ionization in the discharge-tube of radius R and length l , then the resistance will be

$$R_i = \frac{l}{\pi R^2 \sigma} = \frac{l}{\pi R^2} \frac{c^2 m (\omega^2 + n^2)}{e^2 n N_e};$$

the axial current in the tube will be

$$i = \frac{p_1 - p_2 - p_g}{R_i} = \frac{\left(\pi R^2 N l \frac{dH}{dt} - p_g \right)}{l c^2 m (\omega^2 + n^2)} \pi R^2 e^2 n N_e.$$

p_g is due to the loss of the electric force through the glass

walls. The electric field in the glass tube, however, is far from being uniform. If the electric force E between two successive windings in the middle of the coil is strong enough, we might get a pure electrostatic ring discharge in the beginning. And if in general the electrostatic force E_{es} is stronger than the electromagnetic force E_m , we may expect at first an electrostatic discharge, followed by an electromagnetic induction, which cannot fail to appear, when there is sufficient ionization, as long as the electromagnetic laws hold.

If there is no ionization in the tube, the frequency ω_1 of the discharge in the system of fig. 2 will be given by

$$\omega_1^2 = \frac{1}{LC} - \left(\frac{R_1}{2L}\right)^2;$$

if, however, there is a resistance R_i in the tube, the frequency will change, and there is another effect which we have to consider: the capacity of the glass tube will be changed by the motion of the electrons in it. Indeed, if the electric force in the tube is given by

$$E = E_0 \cos \omega t,$$

then

$$m \frac{d^2 x}{dt^2} = E_0 e \cos \omega t$$

is the differential equation of motion of the electron; hence

$$\frac{dx}{dt} = \frac{E_0 e}{m\omega} \sin \omega t.$$

If there are N_e electrons per cm.^3 , then the density of the convection current is equal to

$$i_c = N_e e \frac{dx}{dt} = \frac{N_e e^2 E_0}{m\omega} \sin \omega t,$$

and the displacement current

$$i_d = \frac{k_0}{4\pi} \frac{\partial E}{\partial t};$$

hence the resultant current

$$i = i_d + i_c = \left(\frac{k_0}{4\pi} - \frac{N_e e^2}{m\omega^2}\right) \frac{\partial E}{\partial t} = \frac{k}{4\pi} \frac{\partial E}{\partial t},$$

or the dielectric constant of the tube with the moving ions is equal to

$$k = k_0 - \frac{4\pi N_e e^2}{m\omega^2},$$

or the capacity of the condenser is decreased by ionization

It can be shown by means of Maxwell's equations that

$$\sigma = \frac{kn}{4\pi}.$$

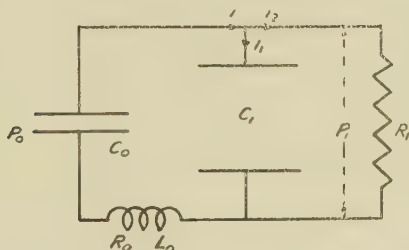
If we take the collisions of the electrons into account, we should expect the following relation:

$$k = k_0 - 4\pi N_e \frac{e^2}{m} \frac{l}{\omega^2 + n^2}.$$

This relation has indeed been given by P. O. Pederson and others.

We shall now consider the oscillations in the system represented by fig. 2. The condenser C_1 is the glass tube with the ionization, *i.e.*, a leaking condenser, or a condenser of capacity C_1 with a parallel resistance R_1 , R_0 and L_0 , are

Fig. 2.



very small resistance and self-induction respectively. We obtain the following equations:

$$i = i_1 + i_2; \quad i = -\frac{dq_0}{dt} = -C_0 \frac{dp_0}{dt}. \quad (1)$$

$$i_1 = -\frac{dq_1}{dt} = -C_1 \frac{dp_1}{dt}; \quad i_2 = \frac{p_1}{R_1}. \quad (2)$$

$$p_0 = iR_0 + L_0 \frac{di}{dt} + i_2 R_1. \quad (3)$$

$$\frac{dp_0}{dt} = \frac{di}{dt} R_0 + L_0 \frac{d^2 i}{dt^2} + \frac{di_2}{dt} R_1. \quad (4)$$

$$-\frac{i}{C_0} = \frac{di}{dt} R_0 + L_0 \frac{d^2 i}{dt^2} + R_1 \frac{di}{dt} - R_1 \frac{di_1}{dt}. \quad (5)$$

But $i_2 R_1 = p_1$; hence from (2) and (3)

$$i_1 = -C_1 \frac{dp_1}{dt} = -C_1 \left[\frac{dp_0}{dt} - R_0 \frac{di}{dt} - L_0 \frac{d^2 i}{dt^2} \right], \quad (6)$$

or, by means of (1),

$$i_1 = \frac{C_1 i}{C_0} + C_1 R_0 \frac{di}{dt} + L_0 C_1 \frac{d^2 i}{dt^2}. \quad . \quad . \quad . \quad (7)$$

Differentiating i_1 , and substituting it in (5) we obtain

$$R_1 C_1 L_0 \frac{d^3 i}{dt^3} + \frac{d^2 i}{dt^2} [R_1 R_0 C_1 - L_0] - \frac{di}{dt} (R_0 + R_1) - \frac{i}{C_0} = 0. \quad (8)$$

In a first approximation we shall neglect the first term and write*

$$\frac{d^2 i}{dt^2} + \frac{di}{dt} \frac{R_1 + R_0}{L_0 - R_1 R_0 C_1} + \frac{i}{C_0 (L_0 - R_1 R_0 C_1)} = 0,$$

which gives the frequency

$$\omega_e = \sqrt{\frac{1}{C_0 (L_0 - R_1 R_0 C_1)} - \frac{1}{4} \left(\frac{R_1 + R_0}{L_0 - R_1 R_0 C_1} \right)^2}.$$

The capacity C_1 is small compared with C_0 , the resistance R_1 is larger than R_0 , and L_0 is a small quantity; the frequency ω_e will therefore be small compared with the frequency

$$\omega = \sqrt{\frac{1}{LC_0} - \left(\frac{R}{2L} \right)^2}$$

of the original system.

In a beautiful experiment with two concentric spheres, Sir J. J. Thomson has shown that when the inner bulb was exhausted it glowed under the action of a discharge; when, now, the outer bulb was exhausted, the ring discharge appeared in the outer bulb and disappeared in the inner one; but as the outer bulb was completely evacuated it ceased to glow, while the inner bulb became luminous again. The extinction of the inner glow is explained by Sir J. J. Thomson as caused by a magnetic screening effect. Indeed, if the current in the ring discharge of the outer sphere is sufficiently strong, its magnetic field, opposite in direction to the field of the exciting coil, may become so strong as to wipe out the magnetic field in the centre. We have repeated this experiment with an electrostatic field alone, and have found results similar to those of Sir J. J. Thomson. The double sphere was placed either between two condenser plates or between two closed and separated loops of wire, parallel to each other over the outer sphere. The inner sphere, properly evacuated, began to glow at first, in the

* A complete solution will be given later.

measure, as the vacuum increased, the outer glow appeared, and finally the inner sphere became luminous again. The distribution of light, however, is different in the two experiments. In the magnetic field the discharge appears mostly in the form of a ring, and quite intense; in the electrostatic field the discharge is a more or less uniform, comparatively weak glow. There is, however, a tendency for two luminous rings to form parallel to the two circular wires. Again, when the outer tube is luminous, the exciting electrostatic field is screened from the inner bulb as the lines of force pass through the outer discharge.

The conditions for the initial discharge under the electrostatic action have been given by Dr. J. Thomson in this Magazine (vol. x. p. 280 (1930)).

Summary.

The electrodeless discharge has been investigated in two parts. In the first part the effect of the magnetic field has been studied on the basis of the theory of Sir J. J. Thomson. It has been shown that the frequency increased by the magnetic ring discharge. The induced ring current and other quantities have been deduced. In the second part the electrostatic effect has been treated. Expressions have been given for the conductivity and the coefficient of self-induction of the glass bulb containing the ionization. The frequency of this discharge has been determined. An experiment analogous to one by Sir J. J. Thomson has been described, showing the effect of electric forces on the glow discharge in a double sphere.

XCI.—*The Effect of Spatial Induction on the Discrimination of Differences in Brightness.* By JAMES F. SHEARER, M.A., B.Sc., Head of the Department of Mathematics and Physics in the Technical College, Coatbridge*.

§ 1. **I** F I is the brightness of a surface and ΔI its least perceptible increment Dr. Houstoun † showed, from the results of König and Brodhun, that when $I/\Delta I$ was graphed as a function of $\log I$ it formed a Gaussian probability curve,

* Communicated by Dr. R. A. Houstoun.

† Phil. Mag. viii. p. 520 (1929).

and in a recent paper *, in conjunction with Dr. Houstoun, I made a further experimental study of the subject. In that paper we gave results obtained by two different methods. In one of these the observer looked into the eyepiece of a telescope and saw two rectangles touching each other. He altered the relative intensity of these by rotating a polarizer. In the other method the observer saw two faces of a rhomb bounded by a sharp line of separation which was brought close up to a circular aperture in a plate; and the relative intensity was altered by means of neutral wedges. The first of these methods proved to be the more accurate, but from the nature of the method the rectangles appeared on a dark ground, *i. e.*, we were restricted to a dark surround. With the second apparatus we were not so restricted, and this paper describes an investigation made with it of the effect of illuminating the surround.

In the apparatus then (*c/.* fig. 1 of former paper) the eye of the observer looked down vertically through an aperture of 2 mm. diameter, which was fitted with a magnifying lens, at the circular aperture in the plate, bisected by the edge of the rhomb and situated about 12 cm. below the magnifying lens. The latter was chosen so that the edge of the rhomb was exactly in focus, while the circular aperture in the plate was of 1 cm. diameter. Thus the eye saw a half shadow field the intensities of which could be altered by moving two wedges. One 220 volt Pearl Osram lamp was used to illuminate both halves of the field, and it was connected in series with a resistance and a milliammeter. By over-running it to 250 volts and reducing the current till the limit of visibility was reached a range of brightness of from 1 to 10^7 was at my disposal. The brightness of the field was determined in absolute measure in the manner described in the previous investigation.

In the first test a piece of cardboard, with a diamond-shaped hole in it, was fixed with its plane at 45° to the vertical and so that the hole was directly above the edge of the rhomb. The cardboard was illuminated by an 8 c.p. carbon lamp which could be moved along a horizontal bench, and the brightness varied through a large range. With this arrangement the eye of the observer saw a diamond-shaped half shadow field divided along its shorter diagonal. This diagonal subtended about $2\frac{1}{2}^\circ$ at the eye, and the diamond appeared in the middle of a rectangular luminous surround which subtended $22^\circ \times 15^\circ$. Outside of this the field was

* *Ibid.* x. p. 433 (1930).

dark. The least perceptible increase in brightness was then determined for various illuminations of the surround ; and the following table gives some results obtained through a Wratten red standard tricolour filter :—

TABLE I.
Log of brightness of test object in "illumination units"
= 3·5

Log of brightness of surround in illumination units.	I/ΔI.
1·85	27·5
0·44	26·5
1·04	46·1
1·65	50·9
2·60	19·9

The illumination unit here used is one-tenth of a photon. The results show that under the conditions of the experiment the power of the eye to discriminate intensity rose to a maximum, and then diminished as the brightness of the surround increased further.

An isolated series of experiments like this is, however, of little value. We know that if the surround is dark $I/\Delta I$ is a probability curve when graphed against $\log I$. We wish to know whether it preserves the same shape when the surround is light, and if so whether it shifts, widens, or increases in intensity. Only when we know the change that occurs over the whole range of brightness can we draw theoretical deductions about the nature of the processes involved. I decided therefore to measure $I/\Delta I$ for red light over the whole range. The conditions thus had to suffer a double variation : (i.) I_0 , the brightness of the surround, had to vary ; (ii.) I , the intensity of the test object, had also to vary.

The cardboard was removed from the 1 cm. aperture and replaced by a brass plate in which a circular hole of $\frac{1}{8}$ inch diameter had been accurately drilled. The angle subtended by the diameter of the test field was thus limited to $1\cdot75^\circ$, while the area of the brass plate to be used as the surround was limited by inserting a short vertical tube immediately below A. The diameter of the surround then subtended 25° .

Before each set of readings was taken the surface of the brass plate was freshly coated with a deposit of magnesium oxide by holding it in the smoke of burning magnesium ribbon. It was decided to work with four brightness levels of the surround, one of these being darkness. The others were obtained by means of three lamps, fixed in clamps at the side, and operated by switches convenient to the hand of the observer. They were measured by bringing the test field up to the same brightness and looking up the current through the Pearl Osram in the calibration chart. The observer was, of course, screened from the direct rays of the lamps and saw nothing whatever but test field and surround.

The procedure adopted was as follows :—After I had been in the dark room for five minutes I set the current through the Pearl Osram to give the lowest intensity desired and kept the surround dark. Then I set the sliding wedges till one-half of the object was just darker than the other, read the wedge-scale with the aid of a 2-volt lamp, set the wedges so that the other half of the object was now the darker, read the scale again, and noted down the readings. Then I set the current to the next higher value, and repeated as above until the maximum current was reached. Next I switched on one of the lamps at the side to get the second level of surround, and repeated for all the current values again until the lowest was reached. Thereupon I switched on the second side-lamp and reascended through the same current values, and, lastly, switched on the third lamp and descended through all the values again. To avoid undue fatigue only nine different intensities of the test field were used, but even this meant 72 settings at one sitting.

The next set was taken the following day at the same time, but, though the surrounds were taken in the same order, the first reading of all was taken at the highest intensity, *i. e.*, the order was descending, ascending, descending, and ascending. The third set was taken with the currents rising and falling as in the first, but the order in which the surrounds were taken was reversed ; and so on, the object being not to favour any one surround in any way, either in always being taken first or always being taken with the currents increasing. Altogether ten sets were made. After the last set was completed the brightness levels of the surrounds were verified, and they were found to have almost their initial values.

It will be noticed that I read the scales myself and also entered up the readings between the settings. This is better than employing an assistant, because it brings the eye into the same condition of adaptation before each reading.

The following table shows the results obtained (the intensities are expressed in illumination units):—

TABLE II.

Log I_0 .	$-\infty$.	1.	2.	3.
Log I.	$I/\Delta I$.	$1/\Delta I$.	$1/\Delta I$.	$1/\Delta I$.
0.0	2.59	2.00	1.95	—
.5	6.56	4.90	2.38	1.96
1.0	17.54	16.69	6.48	2.44
.5	32.26	26.67	17.61	5.83
2.0	37.31	34.25	34.05	15.40
.5	46.30	41.84	35.65	28.99
3.0	44.64	39.59	38.17	41.36
.5	31.65	30.86	33.51	36.05
4.0	20.16	24.92	24.67	25.55

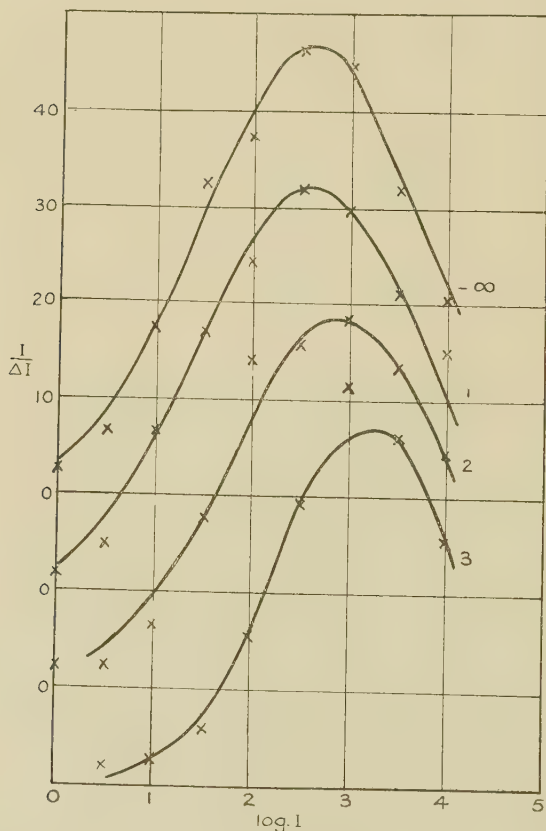
The results are graphed in fig. 1, the smooth curves being exponential curves adjusted by trial and error to fit the observations. In the interests of clearness each curve is displaced one square below its predecessor.

It will be noticed that the smooth curves fit the observations well; but as the level of the surround is raised they are displaced towards the right and at the same time the maximum ordinate diminishes slightly. There is no appreciable change in the shape, *i.e.*, the standard deviation remains unaltered. It will be remembered that the curve represents a distribution of percipient elements with different thresholds; consequently, as the level of the surround is raised the level of the thresholds is raised generally, and simultaneously some of the elements with thresholds below the level of the surround are put out of action. Further than this it would not at present be safe to go; the data are hardly sufficient to establish a formula.

Let us now return to the results recorded in Table I. If we look at abscissa 3.5 in the table we find $I/\Delta I$ increasing slightly as log I_0 changes from $-\infty$ to 3, whereas according to the preliminary investigation it should increase to a maximum at 1.65 and then suffer a marked diminution. The difference must be ascribed to the difference in the size of the test object, which extended beyond the foveal region in the preliminary experiment.

§ 2. The most extensive previous study of the effect of the surround is that of E. M. Lowry*. His surround consisted of a hemispherical bowl 1 metre in diameter, coated inside with matt white paint, which could be illuminated up to 600 millilamberts. The comparison surfaces were aerial

Fig. 1.



images of the faces of an illuminated alabaster prism which were formed in an aperture in the centre of the hemispherical surface. His tables are reproduced below, his Fechner fraction being converted into our form $I/\Delta I$. It will be noted that the angle subtended by the object was $50'$ except in one case, so foveal vision was generally ensured, and that four

* J. Opt. Soc. Amer. xviii, p. 29 (1929); Phot. Ind. xxvii, p. 878 (1929).

different values of surround brightness were employed, viz., 0·0005, 10·7, 53, and 100 millilamberts. Sometimes he used one eye and sometimes two for the observations, as one of the objects of his investigation was a comparison of monocular and binocular vision. In the table I is measured in millilamberts.

His results are in accordance with mine. Some of the sets are irregular, each point resting apparently on a single

TABLE III.—Contrast Sensibility.

0·0005 ml. (50' test field).				0·0005 ml. (2° 30' field).	
Monocular.		Binocular.		Binocular.	
Log I.	I/ΔI.	Log I.	I/ΔI.	Log I.	I/ΔI.
3·85	1·82	3·10	1·16	4·92	1·44
3·93	1·93	3·37	1·50	3·34	3·14
2·02	2·65	3·61	1·97	3·91	5·08
2·81	6·71	2·00	3·10	2·22	6·02
2·86	4·31	2·35	4·83	2·48	10·00
1·53	7·81	2·81	8·00	2·71	12·82
1·97	12·50	1·25	11·49	1·38	9·52
0·66	10·64	1·73	15·38	0·02	20·83
1·88	9·52	0·44	18·18	0·52	47·62
2·18	10·10	1·17	18·87	1·10	125·00
2·44	8·93	1·80	17·24	1·66	52·63
2·78	9·71	2·46	14·29	1·90	83·33
				2·31	41·67
				2·77	28·57

observation ; but if we consider the last four sets, which are averages, the set at 10·7 ml. is represented by the crosses in fig. 2 and the set at 100 ml. by the circles. The two sets at 53 ml. do not differ appreciably from the set at 100 ml. In the diagram I is given in illumination units, the data of Reeves on pupil aperture being used to make the change. The relative heights of the two curves has probably little value, as they were not made under the same conditions, but the shift and change in width are undoubtedly genuine. The maximum moves from 3·3 to 3·8 on the scale of abscissæ when log I_0 goes up from 3·1 to 3·8. According to my

results the maximum changed from 2·6 to 3·2 when $\log I_0$ rose from 1 to 3. It is impossible to state how much of the difference is due to difference in the observers and how much to difference in the conditions under which the measurements were made.

§ 3. After concluding the experiments described in § 1 of this paper it was decided to investigate what effect was produced on the discriminating power of one eye when the other eye was stimulated with the same colour. Accordingly the apparatus was modified as follows:—To the left of the

Fig. 2.



aperture and at the same height was fixed a similar aperture, at a distance equal to the interocular distance of the observer. Through the one aperture the right eye looked at the same half shadow field as before, but now the surround was always kept dark; and through the other aperture the left eye looked at a bright circular patch of the same angular subtense as the surrounds used in § 1. The illumination of the patch was produced by a lamp in series with a resistance, in conjunction with a system of mirrors. By means of various screens fitting closely to the head and nose it was ensured that each eye was acted on only by its appropriate stimulus.

By varying the resistance in the circuit of the second lamp currents were determined which gave brightnesses of 0, 10,

100, and 1000 illumination units, these values being chosen because they were identical with the surround values used in § 1.

The manner of making the test was much the same as before. With the left eye looking at the dark surround the right eye was taken through the same ascending values of intensity as previously, two settings of the wedges being made and noted at each value. Then the current through the second lamp was altered to the predetermined value, so that the left eye now looked at a circular patch of intensity 10 units, and, with this kept fixed, the intensities of the test object were reduced through the range, two settings of

TABLE V.

Patch values.	— ∞ .	1.	2.	3.
Log I.	I/ Δ I.	I/ Δ I.	I/ Δ I.	I/ Δ I.
0.0	2.34	1.78	—	—
.5	3.61	3.08	3.27	3.85
1.0	10.83	9.53	8.03	7.13
.5	17.25	15.40	14.32	17.43
2.0	21.44	20.48	21.87	24.25
.5	23.13	26.23	26.56	27.69
3.0	29.84	26.13	27.23	29.73
.5	29.92	22.64	25.05	29.06
4.0	21.01	20.29	23.68	22.24

the wedges again being made at each point. Next with the circular patch at a brightness of 100 units settings were made for the right eye with the brightness of the test object increasing, and, finally, the brightness of the patch was changed to 1000 units and the test object diminished in intensity through the usual values. All the above readings were taken at one sitting and formed one set. Seven subsequent sets were made with the brightness of the patch and of the test object taken in different orders, and the following table shows the final results. Again the intensities I are in illumination units and the brightness of the patch is expressed in the same units as the values in column 1.

When these results are plotted it is found that *one* error curve can be drawn to fit quite well the values in all the four columns. Thus it seems clear that excitation of the left eye had no appreciable effect on the discriminating power of the right eye. There is certainly no such shift of the curves as has been shown to exist when the brightness of the surround is altered.

During the experiment it was found that sometimes the patch viewed by the left eye disappeared or fluctuated in and out, but naturally when the setting of the wedge was being made I took care that the patch was visible. The fluctuation could be overcome by an effort of will.

In 1930 S. W. Kravkov * investigated the influence of the state of illumination of one eye upon acuity determinations made with the other. The test object consisted of either two white rectangles on a black ground or two black rectangles on a white ground. The separation of these rectangles was variable, and they were viewed by the right eye in an optical arrangement which diminished their size through an artificial pupil. The left eye was either closed or viewed simultaneously a sheet of milk-glass which was illuminated by a 40-watt lamp. It was found that illumination of the left eye increased the acuity of the right eye in the case of black objects on a white ground, but diminished it in the case of white objects on a black ground, that the effect was independent of colour and passed away gradually in three or four minutes if the left eye was closed; also when the peripheral acuity of one half of the one eye was investigated, the illumination of the opposite half of the other eye had no effect.

It is natural to expect that the power of discriminating differences of brightness should vary in the same way as the acuity, and it is consequently difficult to bring Kravkov's work into line with my own.

In one of the papers of the series † of which this forms the concluding number Dr. Houston put forward a quantum explanation of the Purkinje effect. That explanation has now been withdrawn; it would make the colour of a spectral line change slightly with its intensity, and the electromotive force it requires is too large to be possible.

* Archiv f. Ophthalm. cxxiv. p. 76 (1930).

† Phil. Mag. x. p. 431 (1930).

XCII. *The Dielectric Constant and Power Factor of some Solid Dielectrics at Radio Frequencies.* By W. ANDERSON, B.Sc., *Physics Department, The College of Technology, Manchester**.

ABSTRACT.

THE variation of the dielectric constant and power factor with frequency between 150 kilocycles and 1500 kilocycles of some solid dielectrics is investigated.

Introduction.

ALTHOUGH a knowledge of the dielectric constant and power factor of solid dielectrics is of considerable technical and theoretical importance, there appears to be a lack of information published on the subject, and such data as there are by different authors do not seem to be in very close agreement. For example, the B.E.S.A. specification ⁽¹⁾ for ebonite specifies a power factor at 800 cycles for Grade I. of not greater than 0.006, and for Grade II. not greater than .008, Moullin ⁽²⁾ gives "about 0.0125," Warren ⁽³⁾ 0.01 to 0.33, and Griffiths ⁽⁴⁾ gives 0.025 as an approximate value at radio frequencies for the power-loss factor (the product of the dielectric constant and power factor) for best ebonite, and this makes the power factor about 0.008 if the dielectric constant is taken as 3. It has, therefore, seemed desirable to make some further tests on ebonite and other solids, and these are described below.

Preliminary.

The term power factor (herein denoted by $\cos \phi$) as applied to a dielectric means the power factor of a condenser in which that substance is the dielectric, a definition which requires that the power factor of a condenser depends only on the nature of the dielectric and not on the shape of the condenser.

A parallel plate condenser of plate-area A and plate-distance t has a capacity $C = KA/4\pi t$, K denoting the constant of the dielectric.

If the dielectric has a specific resistance ρ the resistance of the condenser is $R = \rho t/A$. At the frequency $f = \omega/2\pi$ the impedance of the condenser is Z with $1/Z = 1/R + j\omega C$.

* Communicated by H. Lowery, Ph.D., F.Inst.P.

So
$$\cos \phi = \frac{1}{R} \frac{1}{\left[\left(\frac{1}{R} \right)^2 + \omega^2 C^2 \right]^{\frac{1}{2}}} = \frac{1}{R\omega C}$$

since $1/R$ is small compared with ωC .

That is,
$$\cos \phi = \frac{1}{R\omega C} = \frac{1}{\frac{\omega \rho t}{A} \cdot \frac{KA}{4\pi t}} = \frac{4\pi}{\omega K \rho}$$

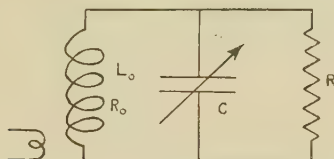
and is independent of A and t .

When an e.m.f. $e = E \sin \omega t$ is applied to such a condenser the mean power absorbed in the condenser is

$$\frac{1}{2} \frac{E^2}{R} = \frac{1}{2} E^2 \omega C \cos \phi = \frac{1}{2} E^2 \frac{\omega A}{4\pi t} K \cos \phi,$$

and so far as the properties of the dielectric are concerned it is proportional to $K \cos \phi$, for which reason $K \cos \phi$ is called the power-loss factor.

Fig. 1.



The chemist or physicist is interested in K and ρ and the engineer in $K \cos \phi = 4\pi/\omega\rho$.

K and ρ depend in general on ω and also on the temperature.

Theory.

Fig. 1 shows an inductance L_0 of resistance R_0 in series with a condenser C and resistance R in parallel. If an e.m.f. $e = E \sin \omega t$ is injected into L_0 the current through L_0 is $i = e/(R_0 + j\omega L_0 + Z)$ and the e.m.f. across Z is $e_1 = Zi = Ze/(R_0 + j\omega L_0 + Z)$, Z denoting the impedance of C and R in parallel.

That is
$$\frac{e_1}{e} = \frac{1}{1 + (R_0 + j\omega L_0)/Z}$$

$$= \frac{1}{1 + (R_0 + j\omega L_0) \left(\frac{1}{R} + j\omega C \right)}$$

$$= \frac{1}{\left(1 + \frac{R_0}{R} - \omega^2 L_0 C\right) + j \left(\frac{\omega L_0}{R} + \omega C R_0\right)}.$$

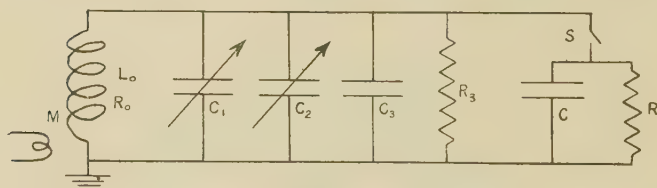
The amplitude E_1 of e_1 is therefore

$$\begin{aligned} \left(\frac{E_1}{E}\right)^2 &= \frac{1}{\left(1 + \frac{R_0}{R} - \omega^2 L_0 C\right)^2 + \left(\frac{\omega L_0}{R} + \omega C R_0\right)^2} \\ &= \frac{1}{(R_0^2 + \omega^2 L_0^2) \left\{ \left[\omega C - \frac{\omega L_0}{R_0^2 + \omega^2 L_0^2} \right]^2 + B^2 \right\}} \end{aligned}$$

with
$$B = \frac{R_0}{R_0^2 + \omega^2 L_0^2} + \frac{1}{R}.$$

If E and ω are kept constant and C is varied the maximum

Fig. 2.



value of $E_1 = E_{1\max.}$ occurs when $C = L_0 / (R_0^2 + \omega^2 L_0^2) = C_0$, say, and is

$$\left(\frac{E_{1\max.}}{E}\right)^2 = \frac{1}{(R_0^2 + \omega^2 L_0^2) B^2}.$$

For a value of $C = C_0 + \delta C$, E_1 is

$$\left(\frac{E_1}{E}\right)^2 = \frac{1}{(R_0^2 + \omega^2 L_0^2) [(\omega \delta C)^2 + B^2]}.$$

Therefore

$$\frac{(\omega \delta C)^2 + B^2}{B^2} = \left(\frac{E_{1\max.}}{E_1}\right)^2 = m^2 \text{ say,}$$

and
$$\pm \frac{\omega \delta C}{\sqrt{m^2 - 1}} = B = \frac{R_0}{R_0^2 + \omega^2 L_0^2} + \frac{1}{R} \quad \dots (1)$$

Experiment.

A diagram of the arrangement employed is shown in fig. 2 in which

L_0 is an inductance suitably chosen for each value of ω .

C_1 is a variable condenser of maximum capacity = 1000 $\mu\mu$ F.

C_2 is a variable condenser of capacity = 0.6 $\mu\mu$ F per angular degree.

C_3 and R_3 represent the effective capacity and parallel resistance of the valve voltmeter.

C and R are the capacity and parallel resistance of the condenser whose dielectric is the material under test. The condenser plates are of brass, circular, and 6 inches in diameter.

S is a switch.

M is a mutual inductance between the test circuit and a valve oscillator.

The deflexion of the galvanometer in the anode circuit of the valve voltmeter is proportional to the square of the applied e.m.f.

The oscillator having been set to a known frequency, C_1 and C_2 are adjusted for resonance and M is adjusted for a suitable maximum deflexion of the galvanometer which is noted. C_2 is varied upwards and downwards to reduce the galvanometer deflexion to half this maximum value, and these two readings of C_2 are noted. If δC_2 is the difference in C_2 in these two settings, then from (1)

$$\frac{R_0}{R_0^2 + \omega^2 L_0^2} + \frac{1}{R_3} = \frac{\omega \delta C_2}{2} \quad \dots \quad (2)$$

since we have made $m^2 = 2$, and the δC_2 in (2) is twice the δC in (1). S is now closed and the procedure repeated.

If $\delta C_2'$ is the new value of δC_2

$$\frac{R_0}{R_0^2 + \omega^2 L_0^2} + \frac{1}{R_3} + \frac{1}{R} = \frac{\omega \delta C'}{2} \quad \dots \quad (3)$$

From (2) and (3)

$$\frac{1}{R} = \frac{\omega}{2} (\delta C_2' - \delta C_2) \quad \dots \quad (4)$$

This gives R , and the only assumption made about the losses in C_1 and C_2 is that they are independent of the settings.

The value of C is obtained from the difference of the readings of C_1 and C_2 with S open and closed. The power factor is then

$$\cos \phi = \frac{1}{\omega C R} = \frac{1}{2} \frac{\delta C_2' - \delta C_2}{C},$$

and K is obtained from

$$\frac{10}{9} \frac{KA}{4\pi t} = C \text{ (expressed in } \mu\mu\text{ F)},$$

A and t being in cm.^2 and cm.

The oscillator was calibrated by means of a heterodyne wave-meter, and the condenser C_1 had been calibrated at the National Physical Laboratory.

The effect of the reaction of the test circuit on the frequency of the oscillator is discussed by Moullin⁽⁵⁾. To test this effect experiments were made at 150 kc. and 1500 kc. in which the magnitude of the oscillator current was varied by adjusting the H.T. volts, and M varied to get the same e.m.f. across C_1 . The normal oscillator current was never less than 20 milliamps., and it was found that decreasing this to 6 milliamps. did not produce any observable difference in δC_2 , from which it is concluded that the effect of frequency drift is negligible here.

The accuracy with which the condenser C_1 can be set controls the accuracy obtained in the determination of K and is here about 1 per cent. A vernier on C_1 would make possible an accuracy of 0.1 per cent.

Since the power factor

$$\cos \phi = \frac{1}{2} \frac{\delta C_2' - \delta C_2}{C},$$

it is seen that the accuracy obtainable for $\cos \phi$ depends mainly on the accuracy in $\delta C_2' - \delta C_2$, which in these experiments is determined to about $\pm 0.1 \mu\mu\text{ F.}$

So if $\cos \phi = 0.005$ and $C = 200 \mu\mu\text{ F.}$, $\delta C_2' - \delta C_2 = 2C \cos \phi = 2 \mu\mu\text{ F.}$, and thus an error of $0.1 \mu\mu\text{ F.}$ in $\delta C_2' - \delta C_2$ involves an error of 5 per cent. in $\cos \phi$. The error is correspondingly less at higher values of $\cos \phi$.

The experimental observations were reduced by plotting the observed values of C and $\delta C_2' - \delta C_2$ against the frequency and drawing smooth graphs through the mean positions. From the smoothed values of C and $\delta C_2' - \delta C_2$, K and $\cos \phi$ were calculated.

Results.

In all twelve samples were tested. In nine of these K and $\cos \phi$ only vary slightly, if at all, in the range explored. The values at the limits of frequency employed are given in the table.

In the remaining three samples the variations in K and $\cos \phi$ are not great, but they are not linear with the frequency and so the relations are shown in the graphs fig. 3.

Curves I. and II. show respectively the dielectric constant and power factor for millboard. Curves III. and IV. show the same quantities for india-rubber, while Curves V. and VI. relate to a specimen of ebonite obtained from a wireless dealer.

TABLE.

	K.		Cos ϕ .	
	150 kc.	1500 kc.	150 kc.	1500 kc.
(1) Glass (mirror).....	6.3	6.3	.0060	.0060
(2) Porcelain (unglazed)	4.7	4.7	.0066	.0066
(3) Ebonite	2.9	2.9	.0077	.0077
(4) Ebonite	3.1	3.1	.011	.011
(5) Keramot	4.0	3.9	.011	.009
(6) Paxolin (Grade "T")	4.2	4.2	.026	.026
(7) Paxolin (Grade "P")	4.3	4.3	.034	.034
(8) Bakelite (Grade EXE 549)	4.7	4.7	.038	.038
(9) Paxolin	5.5	5.1	.044	.049

(3) is an ebonite marked B.E.S.A. 234/1925. This specification requires a power factor of not greater than 0.006 at 800 cycles.

(4) is an ebonite supplied by a manufacturer to the specification "best, quality."

(5) is a loaded ebonite made by Messrs. Siemens Bros., Ltd., Woolwich, London, S.E. 18.

(6) and (7) are samples supplied by Messrs. The Micanite and Insulators Co., Ltd., Walthamstow, London, E. 17.

(8) is a sample of bakelite laminated sheet supplied by Messrs. Bakelite, Ltd., 68 Victoria Street, London, S.W. 1.

(9) is a specimen from a wireless dealer.

According to the theory of Debye (⁶), as applied to polar liquids, K and cotangent ϕ are given by

$$K = \frac{K_1 + K_0 x^2}{1 + x^2}, \quad \cot \phi = \frac{(K_1 - K_0)x}{K_1 + K_0 x^2},$$

where

K_1 = the value of K at $\omega = 0$,

K_0 = the value of K at $\omega = \infty$,

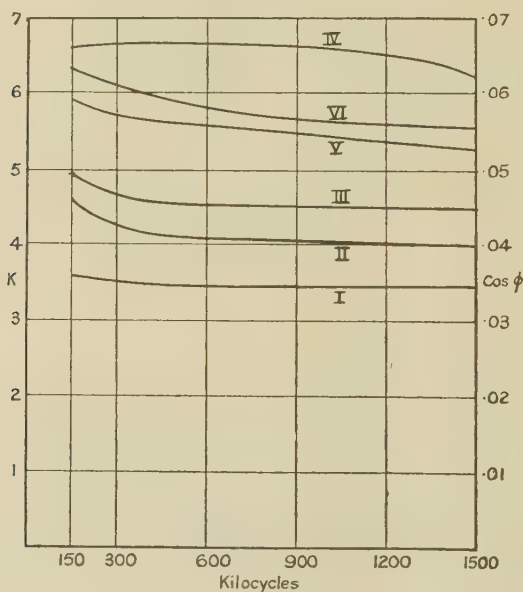
$\omega = 2\pi \times \text{frequency}$,

$$x = \frac{K_1 + 2}{K_0 + 2} \omega \tau = a\omega,$$

τ = "relaxation-time,"

and this is in fair qualitative agreement with experiment. Debye ⁽⁷⁾ also finds that the observations of Errera on ice may be fairly represented by a formula of the same type.

Fig. 3.



An attempt has therefore been made to see if our results for the materials referred to in fig. 3 can be expressed in this way, but without success.

If $\frac{(K_1 - K_0)x}{K_1 + K_0x^2} = \cot \phi \div \cos \phi$, and $\frac{K_1 + K_0x^2}{1 + x^2} = K$,

then $\frac{(K_1 - K_0)a\omega}{1 + a^2\omega^2} = K \cos \phi = K \frac{\delta C_2' - \delta C_2}{2C} = \frac{\delta C_2' - \delta C_2}{2C'}$,

where

$$C' = \frac{A}{4\pi t} \cdot \frac{10}{9} \mu\mu F.$$

is the geometric capacity of the condenser. That is

$$(K_1 - K_0)2C'a \frac{\omega}{\delta C_2' - \delta C_2} = 1 + a^2 \omega^2,$$

and if $\frac{\omega}{\delta C_2' - \delta C_2}$ is plotted against ω^2 the graph should be a straight line, a relation which is not found to hold.

References.

- (1) British Engineering Standards Association Specification, No. 234 (1931).
- (2) E. B. Moullin, 'Radio Frequency Measurements,' p. 315 (Griffin & Co., 2nd ed. (1931)).
- (3) H. Warren, 'Electrical Insulating Materials,' p. 218 (Ernest Benn, Ltd. (1931)).
- (4) W. H. F. Griffiths, 'Wireless World,' April 1930, p. 441.
- (5) *Loc. cit.* pp. 278-281.
- (6) P. Debye, 'Polar Molecules' (Chemical Catalog Co. (1929)).
- (7) *Loc. cit.* pp. 102-108.

XCIII. Light Absorption, the Raman Effect, and the Motions of Electrons in Gases. By V. A. BAILEY, M.A., D.Phil. (Oxon), F.Inst.P., Associate Professor of Physics, The University of Sydney.*

1. **I**N a communication on the motions of electrons in gases which was published † in 1921, the last section contained a discussion of the consistency of our observations with some of the conceptions of the Quantum Theory on the exchanges of energy between electrons and molecules at collisions which were then current.

Since that time the quantum theory has become radically changed, so it is of interest to reconsider its relation to those experimental conclusions of ten years ago.

In order to make the examination wider we shall also consider the results obtained for electronic collisions in some other gases which have been obtained since 1921 by the same or similar methods.

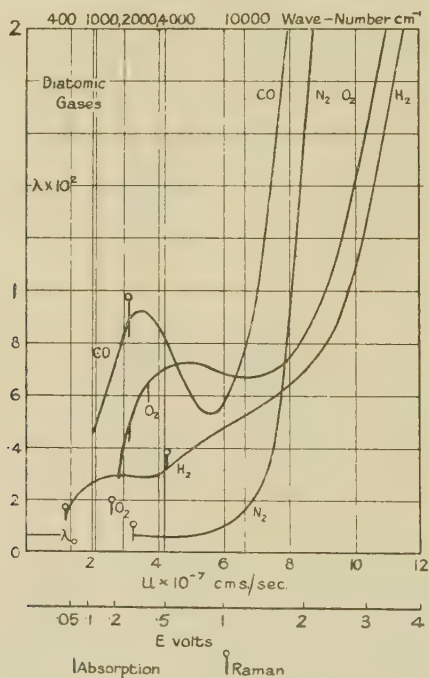
It will be found that the new quantum theory is in closer agreement with the experimental results than was the old.

* Communicated by the Author.

† J. S. Townsend and V. A. Bailey, *Phil. Mag.* (Dec. 1921).

2. The experiments on electrons in gases by means of Townsend's methods (and those developed in Sydney*) determine the values of the mean energy of agitation E (in volts) and the velocity W in the direction of the electric force of the electrons for different values of the ratio Z/p , where Z is the intensity of the uniform electric field and p is the pressure of the gas in which the electrons move.

Fig. 1.



The mean kinetic energy of an electron is $mu^2/2$, where u is the mean velocity of agitation. This energy may be expressed in volts E by the relation $mu^2/2 = eE/300$, e being the charge on the electron in electrostatic units.

The mean free path L may be calculated from the results of the experiments, also the values of the quantity $\Delta E/E$, where ΔE is the mean energy lost by an electron of energy E at a collision with a molecule†.

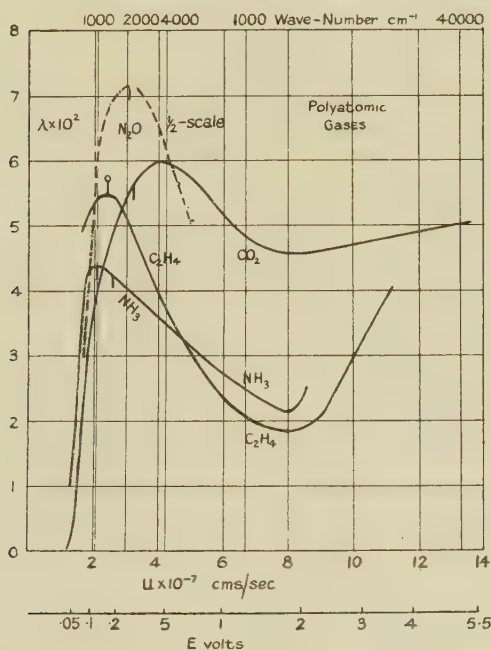
* V. A. Bailey and W. E. Duncanson, *Phil. Mag.* (July 1930).

† *Vide* J. S. Townsend, 'Motion of Electrons in Gases,' 1925 (Clarendon Press).

We are here mainly concerned with the fraction $\lambda = \Delta E/E$, so in respect to the result for L the reader may be referred to a recent comprehensive account by Brose and Saayman*.

The variation of λ as a function of the mean energy of an electron at a collision was shown in the earlier publications by means of tables of the quantity $f = \frac{1}{2}(1+e)$, where e is the coefficient of restitution of the molecule.

Fig. 2.



It is more convenient for our purpose to represent these results by means of curves, as in figs. 1 and 2.

The abscissæ are the values of the velocity of agitation $u \cdot 10^{-7}$ (in cm./sec.), and the ordinates are the values of λ † expressed in "per cent." For our purpose scales are

* H. L. Brose and E. H. Saayman, *Ann. d. Physik*, Bd. 5, p. 797 (1930).

† The values for H_2 have been corrected for the effect of the energy of agitation of the molecules on the energy lost by the electrons at collisions. This correction is negligible for the other gases considered.

also given for the potential E and for the wave-number equivalent to the frequency ν , where ν is derived from E by means of Einstein's relation, $h\nu = Ee$, in which h is Planck's constant. The values of E are given in volts in the figures.

3. In a recent communication * it was pointed out that these curves show that a correspondence exists between the relative magnitudes of the fraction λ and the absorption of light of frequency ν for the same kind of molecule.

Accordingly, for each kind of molecule a short vertical mark is drawn on its λ -curve for the value of λ corresponding to the centre of its *strongest* fundamental absorption band, wherever the data make this possible.

It is evident from figs. 1 and 2, that for each of the molecules CO , CO_2 , N_2O , NH_3 †, C_2H_4 , the maximum of the curve corresponds approximately to the frequency of that radiation in the infra-red region which is absorbed most strongly by the molecule.

The experiments on N_2O are still in progress in Sydney ‡, so the curve representing it is necessarily provisional. With all the other gases the λ -curves had been determined without any suspicion that they might thus be related to the light-absorbing properties of the gases.

The agreement is therefore somewhat remarkable, especially as no attempt had been made to determine the positions of the maxima with the greatest possible precision.

The published data about infra-red absorption in O_2 are somewhat conflicting; *e.g.*, Coblenz § found very weak bands at 3.1 and 4.7 μ , Burmeister || found none, and more recently Snow ¶ mentions a very weak one at 6.5 μ . Coblenz's results are indicated on the curve for O_2 .

The results obtained for the gases HCl , NO , H_2O , and C_5H_{12} are not given here, since the values of λ for these gases have not been determined accurately. The experimental errors are due to the great rate of attachment of electrons to form negative ions, and new investigations

* V. A. Bailey and W. E. Duncanson, *loc. cit.*

† Data for CO and C_2H_4 from Coblenz, *Phys. Rev.* xx. p. 273 (1905).

„ CO_2 from G. Hertz, *Verh. d. D. Phys. Ges.* 1911, p. 617.

„ N_2O „ E. v. Bahr, *idem*, 1913, p. 710.

„ NH_3 „ Coblenz, *loc. cit.*; and Schierkolk, *Z. f. Phys.* xxix. p. 277 (1924).

‡ Skinker and White's observations are not sufficiently extensive for application here.

§ W. W. Coblenz, *Phys. Rev.* xx. p. 273 (1905).

|| W. Burmeister, *Ver. d. Deut. Phys. Ges.* 1913, p. 589.

¶ C. P. Snow, *Phil. Mag.* (Sept. 1929).

are being made with a view to obtaining a higher accuracy. There is a lack of absorption data for C_5H_{12} vapour, but the data for liquid C_5H_{12} show a good agreement with the maximum of the λ curve for the gas.

4. The gases H_2 and N_2 , which are known to be almost completely transparent to infra-red radiation, show no definite maxima.

For the latter, Snow found a very weak absorption band at 4.3μ , which corresponds to the end of the λ -curve, where a slight rise exists. This suggests that more extended observations of λ in N_2 for the low velocities might reveal a maximum.

All the diatomic gases show a very rapid increase of λ with u for the larger velocities, which may indicate that maxima are being approached in the region corresponding to ultra-violet radiation. The fact that the velocities of the electrons are distributed about a mean value u would tend to produce a rise in the curve for values of u distinctly lower than that corresponding to a strong light-absorption band (or notable ionization by collision), but this is not sufficient to account for the parts of the curve for H_2 and N_2 corresponding to values of u less than 5×10^7 .

It is therefore necessary to postulate some other kinds of energy transference from a slow electron to a molecule of H_2 or N_2 in order to account for the fact that λ is several times larger than it would be for perfectly elastic collisions.

5. Another phenomenon which in recent years has required other than light-absorption transference of energy for its full explanation is that known as the Smekal-Raman effect in the scattering of light.

It is therefore of interest to examine whether any connexion exists between the λ -curves of N_2 and H_2 and the frequency differences $\Delta\nu$ between the fundamental Raman lines and the unaltered Rayleigh lines. The values of u given by $\frac{1}{2}mu^2 = h\Delta\nu$ are indicated by short vertical lines with circles attached (to distinguish them from the absorption lines); the values of $\Delta\nu$ are those found by Rasetti*.

With H_2 the Raman lines correspond remarkably with the parts of the λ -curve which are beginning to rise. This is made clearer by indicating with a horizontal line on the diagram the value λ_0 for perfectly elastic collisions between electrons and molecules.

* F. Rasetti, Phys. Rev. xxxiv. p. 367 (1929) Proc. Nat. Acad. Sci. Wash. xv. (1929).

With N_2 this correspondence is perhaps masked by the presence of the weak absorption band found by Snow close to the Raman line, but the λ -curve is definitely beginning to rise at about the velocity 5×10^7 .

Unfortunately, the Raman line for O_2 lies a little outside the λ -curve, but it may also possibly contribute to the rise of the curve in its neighbourhood.

In CO the Raman and absorption lines coincide, in C_2H_4 the strongest are not very different, and in both gases they lie close to the maxima of the λ -curves. It is interesting to note that in C_2H_4 the Raman effect was found by Rasetti to be exceptionally intense.

With N_2O , CO_2 , and NH_3 the strong absorption bands mask any effect which may correspond to a Raman line.

It is probable then, that with polyatomic molecules, the energy transferred from a slow electron to a molecule has a relation to the energy transferred from radiation to a molecule, whether in the process of absorption or of scattering.

In the current theories these processes are referred to changes in the energy of rotation or vibration of the molecule. Accordingly we may refer the larger energy losses of electrons in H_2 to a rotational change when $u = 1.25 \times 10^7$ and a vibrational additional change when $u = 4.28 \times 10^7$.

It is well known that the intensity of the Raman lines increases with the frequency of the exciting light. Something analogous may happen in regard to the probability of certain transferences of energy from the electron to the molecule as u increases, and the rapid rise of the λ -curves in the region of velocities $8-10 \cdot 10^7$ may be in part due to this.

6. If the conclusions about the relation between λ -curves and absorption are correct, it is possible to predict something about the curves for certain gases in which the motions of electrons have not yet been investigated. For example, since, according to Burmeister*, Cl_2 and Br_2 are transparent to radiations of wave-length greater than 1μ , we may expect low values of λ for velocities less than 6.6×10^7 . The fact that these gases are coloured makes it probable that large values of λ will be found for velocities between 7×10^7 and 12×10^7 .

Experiments with these gases are projected in order to examine these points.

* *Loc. cit.*

It would also be interesting to compare the λ -curves for C_2H_4 and HCN, since Burmeister finds a great similarity between the light-absorbing properties of these two gases.

7. For many years Franck and Hertz and their followers have expounded views which are in strong disagreement with many of the experimental conclusions of Townsend and his collaborators.

For example, they have stated that the lowest electronic energies which correspond to large losses at collisions with the molecules of hydrogen and oxygen are 11 and 9 volts respectively, and that with lower energies the losses are essentially due to an "electron affinity" which also causes negative ions to be formed in these gases*.

But in 1926 Franck's † opinion of his original experiments and his arguments on the energy losses of slow electrons in H_2 and O_2 is that they are inconclusive. He is evidently aware of the results for these and other gases obtained by us since 1921, for in the same book he and Jordan say ‡: "In a very indirect way Townsend and Bailey have calculated the effective cross-sections from their experiments on the lateral diffusion of electrons in an electric field. The results agree only in part with those already mentioned. Theory and experiment are not suitable for discussion in the scheme of this book."

Among the methods mentioned in their book is that used by Franck and Hertz § in 1913, and so it is interesting to recall that in that year they confidently asserted "In agreement with Lenard, it is found that the free paths of electrons, with velocities between 10 and 2 volts, is very near to the free path calculated by means of the kinetic theory of gases." This fact, then, may be taken as one reason why Franck and Jordan do not consider our experiments "suitable for discussion."

From the first experiments || made by Townsend's method with air at low pressures and small electric forces, it was shown that electrons move long distances in the gas and

* *E. g.* J. Franck und G. Hertz, *Verh. d. Deut. Phys. Ges.* 1913, p. 932.

† J. Franck und P. Jordan, *Handbuch d. Physik*, Bd. xxii. p. 721 (1926).

‡ Translated from the German.

§ J. Franck und G. Hertz, *Verh. d. Deut. Phys. Ges.* 1913, p. 389.

|| J. S. Townsend and H. T. Tizard, *Proc. Roy. Soc. A*, vol. lxxxviii. p. 336 (1913).

collide with large numbers of molecules of oxygen without becoming attached to the molecules and forming ions.

8. Our experiments of 1921* also showed plainly that, even at pressures of several mm. of Hg, the electronic streams in hydrogen (with energies less than 6 volts) contained no appreciable number of negative ions, and Brose's later experiments on oxygen indicated that notable energy losses occurred without negative ions being evident. Franck and Hertz† and their supporters have maintained that all electrons on reaching a critical potential lose all their energy in a collision with a molecule of a gas, so that when electrons move under electric forces in gases at a few millimetres pressure the critical potential is an upper limit to the energy that the electron may acquire.

The conception that, in general, only a fraction of the collisions with a given energy E result in a definite process is originally due to Townsend, a fact which is recognized by L. Bloch, who stated in his book ‡:

“Par contre, on doit à Townsend diverses constatations expérimentales qui montre la nécessité de faire intervenir dans le calcul des courants d'ionization un *facteur de probabilité* différent de 1.

“Il n'en demeure pas moins qu'un resultat intéressant, tout à fait acceptable pour les theories des quanta, ressort des observations de Townsend; La Dynamique des chocs entre électrons et atoms fait intervenir des facteurs de probabilité spécifiques”

This conception is referred to by Franck§ and Jordan under the name “Ausbeute,” but in discussing it they make no reference to Townsend.

It is of interest to note that Townsend's conception is now incorporated in the new quantum theories of Dirac, Heisenberg, and Schrödinger.

The same conception must also be borne in mind in interpreting the curves giving λ in terms of E . It is only in a small proportion of the collisions with velocities greater than a critical value that electrons lose energy in amounts greater than the average small loss λE .

* J. S. Townsend and V. A. Bailey, *loc. cit.*

† E. g., J. Franck and G. Hertz, *Verh. d. Deut. Phys. Ges.* 1913, p. 932.

‡ ‘Ionization et Resonance des Gas et des Vapeurs,’ 1925.

§ *Handbuch d. Physik*, Bd. xxiii, p. 721 (1926).

XCIV. *Transient Phenomena at the Breaking of an Inductive Circuit.* By J. A. WILCKEN, B.Sc., Ph.D., Lecturer in Electrical Engineering at Armstrong College, Newcastle-upon-Tyne *.

WHEN an inductive circuit on direct current supply is broken the stored electromagnetic energy becomes free and is dissipated as heat in the switch, usually giving rise to an arc between the separating switch-jaws. As long as the arc is burning the current persists, and so a further amount of energy is drawn from the supply until the arc is extinguished. In an alternating current system the problem is essentially different, since the electromagnetic energy is not permanently stored, but oscillating between the source and the circuit and vanishing in the integral over a whole number of periods, as long as the current wave-form is symmetrical.

The decay of direct current, with the concomitant rise in potential at the switch, and the bearing of these phenomena on commutation, have been thoroughly discussed, notably by Arnold and La Cour †.

The question regarding the energy, however, seems to have received less consideration than it merits, in view of its importance in connexion with switch design; in the present paper, attention is mainly directed towards the energy dissipated in the switch, as dependent on the constants of the circuit and the speed of breaking.

We consider a direct current circuit of constant inductance L , and assume the total resistance to increase from R , the resistance of the circuit proper, to infinity, during the period 0 to T . The function $\frac{RT}{(T-t)}$ satisfies this condition for $T \geq t \geq 0$, and may be taken to represent the actual value of the resistance, when the increase is due to a gradually diminishing contact area or pressure at the switch; the contact resistance alone is then

$$\frac{RT}{(T-t)} - R = \frac{Rt}{(T-t)}.$$

When an arc strikes between the switch-jaws the resistance increases as the arc becomes longer, but the exact manner in which this increase takes place is not known. We shall adopt the same expression, as a plausible approximation, also in this case.

* Communicated by the Author.

† 'Theorie der Wechselströme,' i. pp. 620-34 (Berlin, 1923).

It is convenient to take the period of breaking, T , as unit of time, and write simply $\frac{R}{(1-t)}$ for the total resistance.

The voltage equation is then

$$L \frac{di}{dt} + \frac{R}{1-t} = E, \quad . \quad . \quad . \quad . \quad (1)$$

valid for $1 \geq t \geq 0$, and the solution is written

$$i = (1-t)^{\frac{R}{L}} \left\{ \text{const.} + \frac{E}{L} \int (1-t)^{-\frac{R}{L}} dt \right\}. \quad . \quad . \quad (2)$$

Determining the integration constant by the initial condition $i/0 = \frac{E}{R}$, and introducing the time constant $\frac{L}{R} = \tau$, or, measured in units of T , $\frac{\tau}{T} = \theta$, we find, after reduction,

$$i = \frac{E}{R} \cdot \frac{1-t}{1-\theta} \{1 - \theta (1-t)^{\frac{1}{\theta}-1}\}, \quad . \quad . \quad . \quad (3)$$

provided $\theta < 1$. The potential difference at the switch is given by

$$e_1 = \frac{R i t}{1-t} = \frac{E t}{1-\theta} \{1 - \theta (1-t)^{\frac{1}{\theta}-1}\}, \quad . \quad . \quad (4)$$

and at the moment $t=1$,

$$e_1' = \frac{E}{1-\theta}. \quad . \quad . \quad . \quad . \quad (5)$$

If $\theta=1$, the solution is written

$$i = \frac{E}{R} (1-t) \{1 - \log(1-t)\}, \quad . \quad . \quad . \quad (6)$$

to which corresponds a switch voltage

$$e_1 = E t \{1 - \log(1-t)\}. \quad . \quad . \quad . \quad (7)$$

The potential difference at the switch would thus become infinity towards the end of the period of breaking if that period is equal to the time constant of the circuit; the physical interpretation is, of course, that the circuit is not broken, but the arc remains until T has become greater than $\frac{L}{R}$.

The energy dissipated in the switch is given by $\int_0^1 e_1 i dt$, but the general expression for this integral is too complicated

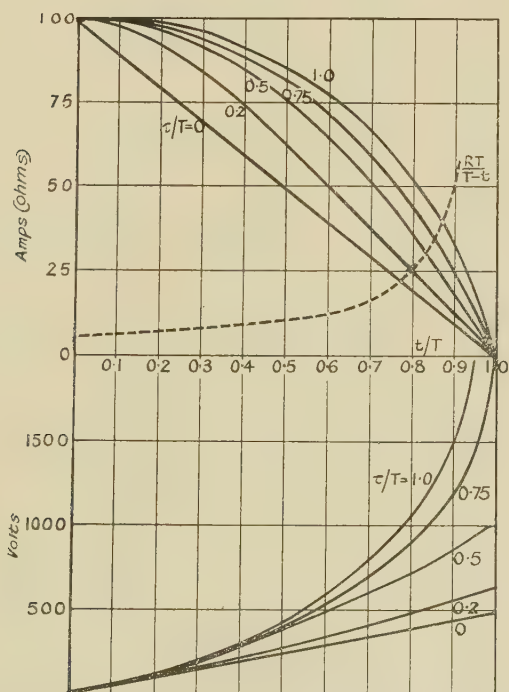
to admit of interpretation, except when the circuit is strictly non-inductive. In this case $\theta=0$, and the current,

$$i = \frac{E(1-t)}{R},$$

decreases linearly, while the switch voltage is simply

$$e_1 = Et,$$

Fig. 1.



Current decrease and switch voltage rise on breaking inductive circuits.

and the energy

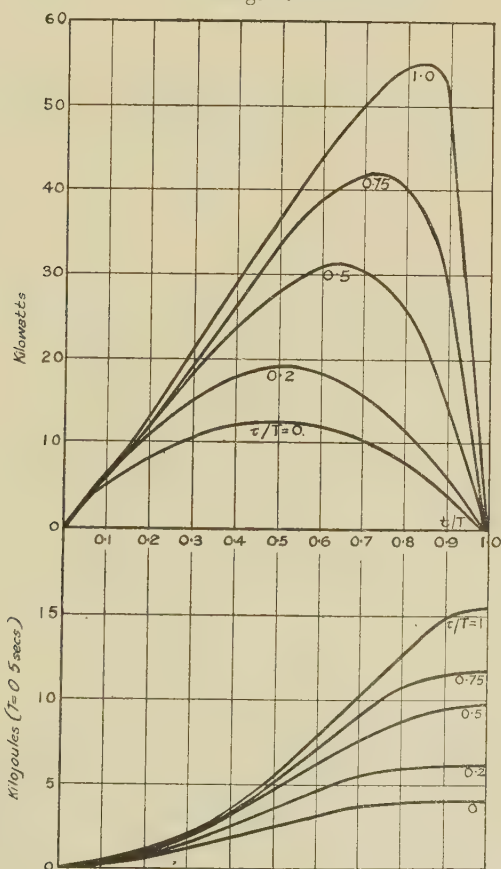
$$\int_0^1 e_1 i dt = \frac{E^2}{R} \int_0^1 t(1-t) dt = \frac{E^2}{6R} \dots (8)$$

To illustrate the behaviour of circuits with a finite inductance I have calculated the following tables, showing the variation of current, switch voltage, and power throughout the time of breaking for different values of the

$t/T =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
(1) $\tau/T=0$.												
Current	100	90	80	70	60	50	40	30	20	10	0	amperes.
Switch volts	0	50	100	150	200	250	300	350	400	450	500	volts.
Power	0	4.50	8.00	10.50	12.00	12.50	12.00	10.50	8.00	4.50	0	kilowatts.
(2) $\tau/T=0.2$.												
Current	100	97.7	91.8	83.3	73.1	61.7	49.7	36.9	25.0	12.5	0	amperes.
Switch volts	0	54.2	114.7	178.5	243.7	308.5	372.8	430.5	500.0	562.5	625.0	volts.
Power	0	5.30	10.53	14.87	17.81	19.03	18.53	15.89	12.50	7.03	0	kilowatts.
(3) $\tau/T=0.5$.												
Current	100	99.0	96.0	91.0	84.0	75.0	61.0	51.0	36.0	19.0	0	amperes.
Switch volts	0	55.0	120.0	196.0	280.0	375.0	480.0	595.0	720.0	855.0	1000.0	volts.
Power	0	5.44	11.52	17.84	23.52	28.12	30.72	30.34	25.92	16.24	0	kilowatts.
(4) $\tau/T=0.75$.												
Current	100	99.2	97.0	93.2	88.1	80.9	71.6	59.7	44.9	26.0	0	amperes.
Switch volts	0	55.1	121.2	199.7	293.7	404.5	537.0	696.5	898.0	1170.0	2000.0	volts.
Power	0	5.47	11.76	18.61	25.87	32.72	38.45	41.58	40.32	30.42	0	kilowatts.
(5) $\tau/T=1$.												
Current	100	99.5	97.9	95.0	90.6	84.7	76.7	66.1	52.2	33.0	0	amperes.
Switch volts	0	55.3	122.4	203.6	302.0	423.5	575.2	771.2	1044.0	1485.0	∞	volts.
Power	0	5.50	11.98	19.34	27.36	35.87	44.12	50.97	54.50	51.20	—	kilowatts.

characteristic ratio $\frac{L}{RT} = \frac{\tau}{T}$. The tabulated figures may readily be interpreted in terms of actual inductances and time units; thus, for example, the values in table (2) $\tau/T=0.2$, would correspond to a circuit of inductance

Fig. 2.



Power and energy dissipated in the switch.

0.25 henry, resistance 5 ohms, which is broken in $\frac{1}{4}$ second. The values are given in ordinary units for a 500-volt supply and a normal steady current of 100 amperes.

Fig. 1 shows the curves of current decrease and rise of potential for five different values of τ/T , and fig. 2 gives the

power dissipated at any moment and the total energy up to any moment released at the switch, assuming $T = 0.5$ second.

The danger in breaking an inductive circuit lies not only in the rise of potential, which may cause puncture of insulation, but also in the increase in power absorbed by the switch. The energy is not released at a uniform rate during the period of opening, but if the switch is opened quickly the power rises to a high maximum value towards the end of the period.

A switch opening a 500-volt circuit of inductance 0.25 henry and resistance 5 ohms in $\frac{1}{15}$ second must dissipate energy at the rate of 40 kilowatts towards the latter third of the time of breaking.

XCV. *A Simple Method for the Numerical Solution of Differential Equations.* By W. G. BICKLEY, D.Sc., Imperial College of Science and Technology*.

METHODS for the approximate integration of differential equations are of considerable value, especially in technical applications, both in cases where the formal solution is unobtainable, and also in cases where, though obtainable, the formal solution is of such form that the calculation of pairs of corresponding values of the variables is troublesome. Some years ago, a useful and comprehensive series of articles upon graphical methods of solving differential equations of the first order was contributed to the 'Mathematical Gazette' by Professor Brodetsky⁽¹⁾. While these methods are of considerable value and interest, especially when it is merely the general trend of the solution that is sought, it will be conceded that if any considerable degree of accuracy is desired, recourse to numerical methods is inevitable. Of numerical methods, that of Runge-Kutta is ingenious, and has the advantage that it allows the use of a fairly large interval—at the expense of a number of separate computations for each interval, and by the use of a formula whose derivation is complicated and which is not easily remembered. The method generally regarded as the best is due to Adams⁽²⁾, and depends upon an integration formula in finite differences. This method has undoubted merits, and for very elaborate work is probably the most accurate, since as many orders of differences can be used as

* Communicated by the Author.

prove to be necessary. However, all exponents of the method advise—for good and obvious reasons—that the interval should be chosen so that the third and higher differences are small. If these differences be neglected, the finite difference integration formula becomes equivalent to Simpson's Rule, and the purpose of the present article is to outline a method in which Simpson's Rule plays the primary role. The advantages claimed for the method are:—

(a) Simpson's Rule is a well-known integration formula, so that no acquaintance with the theory of finite differences is necessary before the method can be understood and used.

(b) The arithmetical operations are considerably simpler and shorter, and the amount of writing considerably less, than in the Adams method.

(c) The method is much better adapted to use with any calculating machine than the Adams, the arithmetical operations consisting (see below) of a multiplication by 4, an addition, and a multiplication by $h/3$ (h being the interval)—this last step is, usually, mental—as opposed to the use of coefficients $1/2$, $1/12$, $1/24$, $3/8$, $19/720$, etc.

The principle underlying the method is, of course, the same as in the Adams method. Given a differential equation

$$dy/dx=f(x, y) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and initial conditions, say $y=y_0$ when $x=x_0$, we have, by integration,

$$y-y_0=\int_{x_0}^x f(x, y) \, dx,$$

and the unknown values of y corresponding to a set of values of x are found by the method of successive approximation, essentially due to Picard⁽³⁾.

Suppose that a set of values of y corresponding to a set of equidistant values of x , x_0 , x_0+h , x_0+2h , . . . , x_0+nh , has been obtained, and the corresponding values of dy/dx calculated by (1): thus

x_0	y_0	y_0'
x_0+h	y_1	y_1'
x_0+2h	y_2	y_2'
—	—	—
$x_0+(n-1)h$	y_{n-1}	y_{n-1}'
x_0+nh	y_n	y_n'

From the trend of these values, a fair approximation to the value of y'_{n+1} can be estimated, and an approximation to y_{n+1} then calculated by Simpson's Rule, for, by this rule,

$$y_{n+1} - y_{n-1} = h(y'_{n+1} + 4y'_n + y'_{n-1})/3.$$

This is now used to correct y'_{n+1} , and another of the above formula gives a second approximation to y_{n+1} . This cycle of operations is repeated until the limit is reached. It is known that, if the integration is carried out accurately, and if the sequence of approximations to y_{n+1} obtained does converge to a limit, then this limit is the accurate value of y_{n+1} . In practice, it is rarely necessary to go beyond the second approximation, since a little judgment, quickened by experience, will usually make the first approximation very close to the true value.

Two further points need mention, the choice of the interval, h , and the determination of a few sets of values to start the process.

The error of Simpson's Rule is of the order

$$h^5 f^{iv}(x, y)/90$$

per double interval, and so is less than

$$h^4(x-x_0) f^{iv}(x, y)/180$$

in the range $(x-x_0)$, f^{iv} being the greatest numerical value of f^{iv} in this range. We must therefore choose h , so that this error will not affect the last place of decimals we desire to retain.

At the beginning, it is necessary that y_1' (at least) be known *accurately*—for errors once introduced are liable to be cumulative in *any* numerical process of this kind. For this purpose the Taylor's series is usually advocated⁽⁴⁾, and is in most cases satisfactory—if, of course, it exists. This may be obtained by evaluating successive differential coefficients by differentiating equation (1) and using the initial conditions, by substitution of a trial series and equating coefficients, or by Picard's iteration method. Which of these involves the least labour depends upon the equation and the initial conditions. Having obtained a few terms of the series, it may with advantage be used until the magnitude of the last term included shows that the accuracy is insufficient—or even beyond, as furnishing approximations to be corrected numerically.

We will now attempt to show the practical working of this method by means of a particular example. The equation we shall use is

$$dy/dx = x + y,$$

with the initial condition

$$y_0=1.$$

The formal solution of this equation is both known and calculable, but this is an advantage, as the accuracy of the numerical work can be definitely checked.

Here, we obtain from our equation,

$$f^{iv}(x, y)=1+x+y.$$

A rough graphical solution shows that, when $x=1$, y lies between 3 and 4, so that if we propose to integrate as far as $x=1$, the value of f^{iv} in our range will not exceed 6. The value of the interval, h , which will ensure the accuracy of four places of decimals must then satisfy the condition

$$6h^4/180 < 0.00005$$

or

$$h < 0.19 \dots$$

By taking $h=0.2$, our purpose would probably be achieved, but it is better to take $h=0.1$, and this we shall do.

The series solution is easily found to be

$$y=1+x+x^2+x^3/3+x^4/12 \dots$$

Putting $x=0, 0.1, 0.2, 0.3, 0.4 \dots$, we find

x	0	0.1	0.2	0.3	0.4
1	1	1.0000	1.0000	1.0000	1.0000
x		.1000	.2000	.3000	.4000
x^2		.0100	.0400	.0900	.1600
$x^3/3$.00033	.00267	.0090	.02133
$x^4/12$.00000	.00013	.00067	.00213
y	1	1.1103	1.2428	1.3997	1.5835

in which the last figure when $x=0.4$ is doubtful. We now tabulate, as in Table I., having material for the first four lines accurate, while the fifth is doubtful.

To correct this, we have

$$\begin{aligned} y(.4) - y(.2) &= 0.1(1.4428 + 4 \times 1.6997 + 1.9835)/3 \\ &= 0.3408, \end{aligned}$$

so that $y(.4) = 1.2428 + 0.3408 = 1.5836$.

We enter the corrected values of $y(.4)$ and $y'(.4)$, but the correction does not affect δy .

We now need a trial value for $y'(.5)$. Noting the successive differences of .21..., .23..., .25..., .28..., we are led

to expect a difference of about $\cdot 32\dots$, and so try $y'(\cdot 5) = 2\cdot 30\dots$ Now, using Simpson, we have,

$$y(\cdot 5) - y(\cdot 3) = 0\cdot 1(1\cdot 6997 + 4 \times 1\cdot 9836 + 2\cdot 30\dots)/3 \\ = 0\cdot 3978,$$

making $y(\cdot 5) = 1\cdot 3997 + 0\cdot 3978 = 1\cdot 7975$,
and $y'(\cdot 5) = 2\cdot 2975$.

TABLE I.

$$y' = x + y ; y_0 = 1.$$

x .	y .	y' .	δy .
0	1·0000	1·0000	
0·1	1·1103	1·2103	
0·2	1·2428	1·4428	
0·3	1·3997	1·6997	
0·4	1·5835 <u>1·5836</u>	1·9835 1·9836	·3408
0·5	— 1·7955 <u>1·7974</u>	2·30... 2·2795 2·2974	·3978 ·3977 ₄
0·6	— 2·0447 <u>2·0448</u>	2·66... 2·6447 2·6448	·4611 ·4606
0·7	— 2·3272 <u>2·3275</u>	3·02... 3·0272 3·0275	·5298 ·5301
0·8	— <u>2·6510</u>	3·45... 3·4510	·6068
0·9	— <u>3·0195</u>	3·92... 3·9195	·6920
1·0	— 3·4363 <u>3·4365</u>	4·43... 4·4363 4·4365	·7853 ·7855

Using this corrected value, we then find 0·3977 instead of 0·3978, as the increment, and so 1·7974 is evidently the value of $y(\cdot 5)$.

For $y'(\cdot 6)$ a trial value is evidently 2·66...,—and so the process goes on. In the table all trials are shown and the

final values are underlined. In practice, trial values would be entered in pencil, and final values in ink, giving one line to each value of x .

Anyone familiar with the use of a calculating machine will realize how simply the multiplication by 4 and the addition can be performed upon such a machine, and the division by 30 mentally, when there will be no writing other than that shown in the table.

As for the accuracy of the results, the solution of our equation, with the given initial condition, is

$$y = 2e^x - 1 - x,$$

so that, when $x=1$,

$$y = 2e - 2 = 3.43656\dots$$

Our error is less than a unit in the fourth place of decimals; what error there is is, in fact, more due to the "rounding off" than to Simpson's Rule, for if the value of $y(1)$ be found by applying this rule to the complete y' column, we find

$$y(1) = 3.43658\dots$$

It is, therefore, usually worth while to carry an extra figure throughout in order to minimize this type of error.

The method leaves nothing to be desired on the score of accuracy, at any rate in this example.

There is no difficulty in applying the same principles to equations of the second—or higher—order, and although the labour is greater, the proportionate saving over that involved in the Adams method is greater still. In the case of the most general equation of the second order, .

$$y'' = f(x, y, y'),$$

we must estimate first approximations to both y and y' at each stage, and then correct them by numerical integration. A cycle now involves the calculation of y'' from the approximations to y and y' , by means of the above differential equation, correcting y' , and then y by Simpson, and it will be appreciated that any method which ensures that our first approximations are close, and thus reduces the number of cycles that are necessary at each stage, will be very desirable.

Such a method, based upon the first and second differences of y , which has worked very well in practice, will now be outlined. It may first be emphasized, however, that it demands very little acquaintance with finite differences to appreciate the fact that the regularity of these affords a check upon the accuracy of the work. In constructing the table

of differences for our purpose, the "horizontal" form is most convenient. This is defined by the scheme

$$\begin{array}{ccc}
 y_0 & & \\
 y_1 & \Delta y_1 & \\
 y_2^2 & \Delta y_2 & \Delta^2 y_2 \\
 \hline
 y_n & \Delta y_n & \Delta^2 y_n
 \end{array}$$

where the rule for the calculation of any difference is

$$\Delta^{m+1}y_n = \Delta^m y_n - \Delta^m y_{n-1}.$$

TABLE II.

$$y'' + y'/x + y = 0; \quad y_0 = 1, \quad y_0' = 0.$$

$x.$	$y''.$	$y'_n - y'_{n-2}.$	$y'.$	$y_n - y_{n-2}.$	$y.$	$\Delta y.$	$\Delta^2 y.$
0.0	-.5000	—	.0000	—	1.0000	—	—
0.2	-.4925	—	-.0995	—	.9900	100	—
0.4	-.4704	-.1960	-.1960	-.0396	.9604	296	196
0.6	-.4342	-.1872	-.2867	-.0760	.9120	484	188
0.8	-.3853	-.1728	-.3688	-.1141	.8463	657	173
1.0	-.3252	-.1534	-.4401	-.1468	.7652	811	154
1.2	-.2559	-.1295	-.4983	-.1782	.6711	941	130
1.4	-.1795	-.1010	-.5420	-.1984	.5668	1043	102
1.6	-.0992	-.0715	-.5698	-.2157	.4554	1114	71
1.8	-.0170	-.0395	-.5815	-.2262	.3400	1154	40
2.0	+.0644	-.0069	-.5767	-.2315	.2239	1161	7
2.2	.1423	+.0255	-.5560	-.2296	.1104	1135	-26
2.4	.2141	.0565	-.5202	-.2213	.0026	1078	-57

With this arrangement, we have the approximate formula

$$\Delta^2 y_n \doteq h^2 y''_{n-1},$$

so that the second difference of y in the next line can be calculated very closely from the value of y'' already known, in the line which has just been completed. Again, we have, approximately

$$y'_{n+1} - y'_{n-1} \doteq 2h y''_n,$$

and this enables us to forecast, with fair accuracy, the next value of y' . With the help of these two formulæ, it is possible to make the process comparatively speedy.

To illustrate, we give in Table II. the final values obtained in integrating Bessel's equation of zero order,

$$y'' + y'/x + y = 0,$$

subject to the conditions that, when $x=0$, $y=1$, and $y'=0$, by this method. An interval of 0.2 is used, and the calculations are carried far enough to enable an approximation to the first root to be obtained. In the light of the foregoing, the table should be self-explanatory, but the reader should compare the values in the column headed $\Delta^2 y$ with 0.04 (h^2) times those in the preceding line of the column headed y'' , and he will see how little adjustment the values of y have needed.

Applying Newton's method to the last line gives, as an approximation to the root,

$$x = 2.405 (0) \dots$$

as compared with the accurate value

$$x = 2.404 (8) \dots$$

By comparison with a table of Bessel functions, it will also be seen that y and y' are each correct to four places of decimals, at each stage.

It may further be noted that the differences show up an error sooner, and with more insistence, than they do in the Adams method. To see this, imagine that an error of amount α creeps into y'_n , by inaccuracy of addition or otherwise. The effect upon y_n is then $h\alpha/3$, but that on y_{n+1} is $4h\alpha/3$ (together with "second order" effects). There will also be, in the case of a second order equation, an error of α in y'_{n+2} , and thus an error of amount $2h\alpha/3$ in y_{n+2} . In consequence of the inequalities of these errors in the y 's, the second differences will at once commence, and will continue, to oscillate about a mean which approximates to $h^2 y''$, so that the presence (and amount) of error will be immediately visible. In the Adams method, the error once introduced remains, apart from "second order" effects, constant, so that only a few differences are affected. This may easily pass unnoticed, since the oscillations soon die out.

Finally, although the equations used by way of illustration have been linear, the method is equally applicable to non-linear equations. It has been used, for instance, to obtain solutions of equations of the form $d^2 y/dx^2 = \alpha \sin y - x \cos y$.

References.

- (1) *Math. Gazette*, ix. and x. (1918-20).
- (2) 'Theories of Capillary Action,' Cambridge, 1883. See also Whittaker and Robinson, 'Calculus of Observations,' or J. B. Scarborough, 'Numerical Mathematical Analysis.'
- (3) *Journ. de Math.* (4) vi. pp. 197-210 (1890); *Traite d'Analyse*, ii. p. 337 (1905).
- (4) E. B. Moulton, in his article in the *Smithsonian 'Mathematical Formulæ and Tables of Elliptic Functions,'* advocates a process of trial and error, which seems cumbersome, and uses Simpson's Rule as a check upon the early values, but abandons it later.

XCVI. *Rates of Condensation and Evaporation in intensively dried Systems. The Effect of Intensive Drying on the Accommodation Coefficient of Liquid and Solid Surfaces for Molecules of their own Vapours.* By F. J. WILKINS, *M.Sc., Ph.D.**

SINCE the early classical investigation of Baker of the influence of intensive drying on the physical properties of substances much research work has been carried out, and extraordinarily diverse and contradicatory results have been obtained. In spite of the controversial nature of many of the experimental data one important difference between dried and undried systems appears to be well established (see, *e. g.*, J. W. Smith, *Phil. Mag.* viii. p. 380 (1929)). This is the slowness with which intensively dried systems return to a condition of equilibrium after having been displaced from it. A particular case of this phenomenon is the decrease observed in the rate of evaporation and condensation during the progress of drying. In the following pages it is proposed to discuss this topic and show that one must conclude that intensive drying decreases the accommodation coefficient of liquid and solid surfaces for molecules of their own vapours.

Rodebush and Michalek (*J. A. C. S.* li. p. 748 (1929); *Proc. Nat. Acad.* xiv. p. 131 (1928)) have shown that the intensive drying of ammonium chloride, while leaving the vapour pressure and density unchanged, causes a marked decrease in the rate of condensation and evaporation. For example, a distillation of ammonium chloride from a hot to a cold bulb was complete for the undried salt in ten minutes; the same distillation for the dried salt took several hours.

* Communicated by Prof. F. G. Donnan, C.B.E., M.A., LL.D., Ph.D., D.Sc., F.R.S.

The decrease in the rate of evaporation due to drying was also noticed in experiments designed to measure the vapour density by the Knudsen effusion method. J. W. Smith (*loc. cit.*) points out that the results of A. Smith and Menzies (*Z. Phys. Chem.* lxxvi. p. 713 (1911)) on the vapour pressure of dry calomel reveal the same effect. They found that dry calomel had no vapour pressure at temperatures at which ordinary calomel had a measurable vapour pressure. He suggests that since the experiments only lasted for times of the order of a quarter of an hour the dry calomel had not vaporized to any extent. Both Smith, and Rodebush and Michalek attribute many of the anomalous vapour density measurements of intensively dried substances to the slowness with which they vaporize.

Similar experimental evidence has been obtained by J. W. Smith for liquid systems. He has demonstrated that drying decreases the rates of distillation of nitrogen peroxide (*J. C. S.* cxxvii. p. 869 (1927)), and ethyl bromide (*ibid.* cxxxi. p. 2573 (1931)) considerably. The most detailed measurements were made with ethyl bromide. In this case it was shown that as drying proceeded the rate of distillation gradually decreased (the rate of decrease with time of drying being greatest in the initial stages), until, after twenty-six weeks' drying, the rate of distillation became almost constant, the final rate being half the initial. Moreover, during drying no change took place in the vapour pressure of the ethyl bromide. It was concluded that intensive drying did not alter the equilibrium of the liquid phase.

The above experimental results may be extended quite readily, for it can be shown that they indicate a decrease in the accommodation coefficient as drying proceeds. The results of Rodebush and Michalek will be discussed first.

1. *The Rates of Vaporization and Condensation of Ammonium Chloride.*

The distillation experiment already referred to was carried out under conditions such that the temperature of the cold bulb was so low that the vapour pressure of the condensed ammonium chloride was probably too small to cause any appreciable back diffusion into the hot bulb. The rate of distillation, therefore, corresponds with the rate of vaporization. Let V be the rate of vaporization from unit area of the solid surface at a given temperature, and P the vapour pressure at the same temperature. Then if α is the accommodation coefficient, the rate of condensation on to unit area

from the vapour when it is in equilibrium with the solid is $k\alpha P$, where k is a constant given by the Herz-Knudsen equation as equal to $43.07.10^{-7} \sqrt{M/T}$. M =mol. wt., T =abs. temp. Therefore

$$V = k\alpha P.$$

Now experiment shows that V decreases with drying, while the vapour pressure and density are unchanged. Therefore k and P are unchanged by drying, and the decrease in V must be accompanied by a decrease in α : i.e., intensive drying causes a decrease in the accommodation coefficient of ammonium chloride vapour at an ammonium chloride surface. α appears to be decreased by a factor of about ten.

2. The Rate of Distillation of Ethyl Bromide.

In these experiments the receiver was kept at a temperature of 0°C ., where the ethyl bromide has a considerable vapour pressure. Let V , k , α , and P have the same meaning as before, and let p be the vapour pressure of ethyl bromide in the still at a distance from the surface of the liquid equal to a mean free path of a molecule in the vapour phase. Then, D , the rate of distillation when a steady state has been established, is equal to the difference between the rates of vaporization and condensation at the ethyl bromide surface. The molecules striking the surface come from a distance into the vapour phase of a mean free path, and the rate of condensation is then

$$k\alpha p.$$

Therefore,

$$D = V - k\alpha p. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Further, since P is the vapour pressure,

$$V = k\alpha P \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and consequently,

$$D = k\alpha(P - p).$$

Now P is unaltered by drying; further, since p is a function for given experimental conditions only of P and the vapour pressure in the receiver, it also is unaltered by drying. The experimentally observed decrease in D must therefore be due to a decrease in either k or α separately or together. It is quite inconceivable that k should be decreased by drying, for that would imply (*vide supra*) a decrease in the molecular weight of the vapour. We must therefore conclude that intensive drying causes a decrease in the accommodation

coefficient of ethyl bromide molecules at an ethyl bromide surface. Since D is proportional to α , and since D is decreased to $\frac{1}{2}$ by drying, α must be decreased by a half.

A second conclusion which may be drawn is that the rate of vaporization is also decreased by intensive drying. This follows immediately from the equation for equilibrium

$$V = k\alpha P,$$

since both k and P are unaltered by drying. These results are, therefore, in accord with those of Rodebush and Michalek, since the rates of vaporization and condensation of both solid ammonium chloride and liquid ethyl bromide are decreased by intensive drying.

One of the features of these experiments which distinguishes them sharply from many others on intensive drying is that the effects become observable after comparatively short periods of drying. This perhaps demonstrates most effectively how sensitive rates of condensation and evaporation are to traces of water, for the *total* amount of water removed was not sufficient to cause a measurable change of vapour pressure.

Discussion.

Relevant experimental determinations of α .—The number of determinations of the accommodation coefficients of solids for condensing gas molecules is not large. Only in one or two instances have estimates been made of the accommodation coefficient of a solid for its own vapour. Volmer and Estermann (*Zeits. f. Physik*, vii. p. 1 (1921)) state that near the melting-point α for mercury against mercury is slightly less than unity and increases as the temperature is lowered. Langmuir (*Phys. Rev.* ii. p. 329 (1913)) in his work on the vapour pressure of high melting metals suggests that α is unity.

Only in two investigations has the influence of impurities on the accommodation coefficient been studied. Roberts (*Proc. Roy. Soc. A*, cxxix. p. 146 (1930)) has measured the accommodation coefficients of helium on wires of nickel and tungsten. He finds that α for a freshly outgassed wire is in both cases less than α for a wire which has been allowed to remain in the apparatus and readorb any residual impurities. These results are in agreement with those deduced above for ammonium chloride, since intensive drying may be considered as a process of removing impurities of a certain type. Johnson (*Proc. Roy. Soc. A*, cxxviii. pp. 432, 444 (1930)) has shown that if α is 0.26 for hydrogen at a glass surface

containing a monomolecular layer of adsorbed water-vapour, then washing with hydrogen *in vacuo* increases α to 0.305. The value of α for a glass surface covered with a layer of hydrogen atoms is 0.295. These figures are completely opposite in type to those of Roberts, and those deduced in this paper, for the removal of the adsorbed water *increases* α .

No measurements of the influence of impurities on α for a liquid/gas interface have been made. Knudsen (*Ann. d. Physik*, xlvii. p. 697 (1915)) finds α for mercury at a liquid mercury surface to be within 1 per cent. of unity. Similarly Alty and Nicoll (*Canadian J. Research*, iv. p. 547 (1931)) find α for carbon tetrachloride and benzene against their own vapours to be nearly unity. None of these liquids had been dried.

Theoretical treatment of α .—It is a well-known fact that α is increased by increasing the roughness of the surface. This is usually interpreted to mean that incident molecules tend to become entrapped in the irregularities and therefore make more than one collision before they leave the surface. If this were the explanation of the decrease in α due to intensive drying one would have to assume that drying tended to make surfaces smoother. For solids this is unlikely and for liquids it is quite impossible.

An extensive theoretical treatment of α has been given by Baule (*Ann. d. Physik*, xlv. p. 145 (1914)), who assumes that the collisions between incident molecules and the surface atoms are perfectly elastic. He shows that

$$\alpha = (1 - \beta\delta),$$

where

$$\beta = \frac{M^2 + m^2}{(M + m)^2},$$

M is the mass of the solid atom and m is the mass of the condensing atom; δ is the fraction of the incident molecules which make only one collision with the surface. The remainder $(1 - \delta)$ make sufficient collisions to come into thermal equilibrium with the surface. Johnson (*loc. cit.*) finds that his results satisfy Baule's equation. He assumes that the "impurities" present on his glass surfaces, water and hydrogen atoms, load the surface vibrators. Using the same method for ammonium chloride and putting δ equal to unity, we obtain the greatest possible effect which the weighting of ammonium or chloride ions with water could have on α . In this way for the dry ammonium ion α is 0.37, while for the ammonium ion weighted with water α is 0.48. The

corresponding figures for the chlorine ion are 0.48 and 0.50 respectively. The observed experimental decrease is of the order of ten times this.

Roberts (*loc. cit.*) also finds that Baule's equation does not correspond with his experimental figures. He concludes that there is an important amount of specular reflexion of the type studied by Stern and his co-workers. This specular reflexion is a direct consequence of the wave-structure of matter, the solid surface acting as a diffraction grating. The condition for reflexion (see Fraser, 'Molecular Rays,' p. 82, *et seq.*), is that $h \sin \theta \sim \lambda$ where θ is the glancing angle, h the average height of the inequalities on the surface, and λ the wave-length of the material particle as given by the de Broglie equation. Impurities will tend to distort the grating and thus spoil the lattice. Consequently the amount of specular reflexion is decreased and α is increased, as his experiments show. Since he used helium the amount of specular reflexion may be considered to be perhaps important, but as θ for ammonium chloride is less than 2° the specular reflexion must be very slight. Roberts's theory, therefore, cannot apply to the cases discussed here.

The accommodation coefficient has been treated theoretically by Langmuir (Phys. Rev. viii. p. 149 (1916)). He suggests that Baule's analysis can have no general validity, as it fails to take into account the attractive forces which exist between molecules and surfaces. The part of his theory which is of importance for this investigation is that he shows that the surface field, which is effective in the condensation process, must be considered as that due not only to the atom at which condensation is occurring, but also to those which surround it. The effects discussed in this paper seem to indicate that a polar impurity such as water may influence surface fields over distances of several molecular diameters. (The transference of polar influences over comparatively great distances has been advocated by de Boer—see, *e. g.*, de Boer and Zwikker, *Zeits. Phys. Chem. B.* iii. p. 407 (1929), and Topley, J. S. C. I. (1931).) Whether Langmuir's theory is able to account for the results quantitatively it is not possible at present to determine. An analogous problem, the reflexion of electrons at a surface has been treated by Nordheim (*Zeits. f. Physik*, xlvi. p. 833 (1928)) using the new quantum mechanics.

The influence of intensive drying on the rate of evaporation has been discussed by J. W. Smith (*loc. cit.*). It is interesting to note that all changes in the rate of evaporation are exactly balanced by changes in the accommodation

coefficient, since the vapour pressure remains unchanged (*cf.* equation 2).

Summary.

It is shown that the experiments of Michalek and Rodebush which indicate that intensive drying decreases the rate of evaporation and condensation without changing the vapour pressure, prove that intensive drying decreases the accommodation coefficient of ammonium chloride for its own vapour. J. W. Smith found that drying decreased the rate of distillation of ethyl bromide without altering its vapour pressure, and it is shown that a similar conclusion can be drawn. These conclusions are discussed theoretically.

My thanks are due to Dr. N. K. Adam and Dr. J. W. Smith for opportunities of discussion, and to Prof. F. G. Donnan, F.R.S., for his continued interest in my investigations.

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XCVII. *Precision Measurements of the Crystal Parameters of some of the Elements.* By Prof. E. A. OWEN, M.A., D.Sc., and JOHN IBALL, M.Sc., University College of North Wales, Bangor*.

[Plates XVII. & XVIII.]

IN the course of an investigation on the accurate determination of the crystal parameters of certain alloys and the effect of different methods of heat treatment on these alloys, we had occasion to measure the crystal parameters of a few of the elements to a high degree of accuracy. The parameters of most of the elements contained in published tables† are given to the third significant figure, but for the work in hand it was desired to know the parameters, if possible, to the fourth figure. As the few elements which were initially examined yielded consistent results to this accuracy, it was decided to extend the measurements to include as many of the elements having

* Communicated by the Authors.

† See 'International Critical Tables,' vol. i, p. 340 (1926).

cubic structure as were immediately obtainable. The results of the measurements are collected together in this paper, which is confined entirely to elements possessing cubic symmetry. It may conveniently be divided into two sections, namely: (1) the examination of the elements aluminium, copper, and silver by a precision method using radiations of different wave-lengths; (2) the examination of a number of other elements by the same precision method employing only copper radiation.

Method of Measurement.

The X-ray camera employed for the measurements was similar to that used by Gayler and Preston* in an investigation on the age-hardening of aluminium alloys. It consists of a cylindrical drum in the circumference of which is a narrow slit. A divergent beam of X-rays from the slit falls on a specimen of the material under examination, which is mounted on a curved plate covering a short arc diametrically opposite to the slit. After reflexion the rays fall on a photographic film extending over half the circumference of the drum and arranged symmetrically about the slit. With this arrangement the rays come to a sharp focus on the film and lines are observed in symmetrical positions on either side of the slit. From a knowledge of the distance (s) between corresponding lines and the radius (r) of the drum, the angle of reflexion (θ) can be calculated from the relation $s/8r = (\pi/2 - \theta)$. Combining this with Bragg's law of reflexion, the distance between the planes giving the reflexion is given by the relation $d = \lambda/2 \cdot \sec s/8r$, where λ is the wave-length of the radiation employed. Since $d = a \cdot f(hkl)$ for cubic crystals, where a is the lattice parameter and h , k , and l , the Miller indices of the reflecting plane, then $\delta a/a = \tan s/8r \cdot \delta(s/8r) = (s/8r)^2 \delta s/s$, where $s/8r$ is small. The percentage error in measuring s is therefore multiplied by the square of a small quantity, so that the percentage error in the value of the parameter is exceedingly small.

It will be seen from the photographs that when the reflecting angle θ is large the lines lose in definition, making it more difficult to measure the distance between them with a high degree of accuracy. High accuracy of measurement is, however, not necessary at large angles of reflexion. For instance, the angle of reflexion of the $K\alpha$ -radiation of copper from the (422) planes of rhodium is about 83° . Now $\delta\theta = \delta s/8r$, so that if $r = 50$ mm. and the error in measuring

* Journ. Inst. Metals, xli. (1929).

s is as much as ± 0.3 mm., then $\delta\theta = 3/2000$ and $\delta a/a = 0.02$ per cent.

The camera was carefully constructed, and certain modifications were introduced to increase the accuracy of the determinations. The drum had to be perfectly rigid, so that it remained undistorted when subjected to strains such as those set up by the weight of the lead shield covering the photographic film. The final form of camera adopted was made in the following manner:—

Two cast iron disks about 4 inches in diameter and $\frac{3}{8}$ inch thick were turned in the lathe to form two annular rings about $\frac{3}{4}$ inch wide. The outside of the rings was carefully turned, so that a true surface at right angles to the plane of the rings was obtained and the rings were both of exactly the same diameter. In a brass sheet $\frac{1}{16}$ of an inch thick and $1\frac{3}{4}$ of an inch wide were cut suitable slots to be covered by the photographic film and by the material to be examined. The sheet was bent into a circle in the rolling machine, so as to fit over the two circular rings already prepared, and was fixed to the rings with about a dozen countersunk screws. Constructed thus the camera was extremely rigid. After the screws had been inserted, the camera was put in a lathe and the outer surface made cylindrical by turning off a thin layer of the metal. Thus there could be no possibility of irregularities on the brass surface or of distortion in the camera; and the diameter of the camera could be accurately determined. Several methods were employed to measure the diameter, all of which agreed well with each other, giving for the diameter a mean value of 102.92 mm.

Further precautions had to be taken to obtain the correct radius for the calculation of the crystal parameters. The film was wrapped in black paper and held firmly and uniformly against the circumference of the camera. With this arrangement the film was displaced from the true circumference. Also, since the gelatine of the film was coated with emulsion on both sides, there were two images formed, which were not exactly coincident. The lines were measured from the middle of one to the middle of the corresponding line on the opposite side of the centre of the film. This procedure gave the correct distance between the lines if the images had been formed at the position occupied by the middle of the gelatine between the two emulsions. Hence a correction had to be applied both for the thickness of the black paper and for the position of the images on the surface emulsions of the film. The thickness of the film before development was 0.23 mm., so that the

middle of the gelatine was 0.12 mm. radially from the outer surface of the black paper which was 0.08 mm. thick, and thus was situated on a circle of radius 51.66 mm. The formula from which the results are calculated is derived on the assumption that the reflecting material and the film are on the circumference of the same circle, so that the reflecting material must also lie on the circumference of a circle of radius 51.66 mm. This was arranged by fixing to the camera, around the slot where the reflecting material was placed, a sheet of aluminium which had been carefully rolled down to a thickness of 0.20 mm.

Several slit systems were tried; the arrangement which yielded spectral lines showing the best definition, consisted of a short brass rod through which was drilled a small round hole about 1 mm. in diameter. The rod was covered on the inner face with a thin lead disk, in which was pierced a smaller hole with a needle-point. The rod could be adjusted to any desired position in a tube fixed to the circumference of the camera. Several photographs of aluminium were taken with the slit in different positions and the rod finally fixed in the position which gave lines showing the best definition.

The film was held in position by an elastic band, which also held the reflecting powder in position. Underneath the elastic band and just covering the film was a light steel spring, which acted as a screen and also served to press the film tightly against the camera. Over the elastic band was a lead screen, held in position by two buttons fixed to the camera. The whole camera was mounted on a stout piece of brass, by means of which it was held firmly in a suitable stand.

The film on development contracts. This contraction was very small, but was allowed for by recording on each film, near its ends, two fiducial marks, the exact distance between which was known and assuming a linear contraction along the film. The travelling microscopes were also calibrated against standard lengths and were correct along the scale to within 0.05 mm., an accuracy much higher than was required for the measurement of the film.

The installation consisted of a modified Shearer tube operated on a Schall transformer at a voltage of about 40,000 volts and 10 ma. The exposures varied between two and four hours.

To obtain well-defined spectral lines at these large angles of reflexion, it was necessary, with most of the elements examined, to pay particular attention to the heat treatment.

The metals were annealed *in vacuo* for many hours at appropriate temperatures, varying with different materials, and then quenched in cold water.

Single-coated photographic films were experimented with, but no better definition was obtained whilst the exposures had to be prolonged. A second precision camera having effective radius 69.63 mm. was constructed to serve as a check on the first. Both cameras were used in the investigation.

TABLE I.—Aluminium.

Target.	Arc (mm.).	λ .	Reflecting planes.	Parameter.
				Å.
Chromium.	197.9	Ka_1 }	(311)	4.0415
	195.1	Ka_2 }		4.0415
	113.1	Ka_1 }	(222)	4.0408
	108.4	Ka_2 }		4.0409
Cobalt.	152.3	Ka_1 }	(331)	4.0411
	147.7	Ka_2 }		4.0408
	109.9	$K\beta$	(422)	4.0403
	86.2	Ka_1 }	(420)	4.0403
	77.5	Ka_2 }		4.0398
	152.2	Ka_1 }	(331)	4.0408
	147.9	Ka_2 }		4.0408
	110.1	$K\beta$	(422)	4.0405
	86.2	Ka_1 }	(420)	4.0403
	77.8	Ka_2 }		4.0401
Copper.	207.1	Ka_1 }	(422)	4.0419
	203.3	Ka_2 }		4.0417
	84.4	Ka_1 }	{ 511 } { 333 }	4.0406
	74.6	Ka_2 }		4.0406
Iron.	184.9	$K\beta$	(331)	4.0411
	165.7	Ka_1 }	(400)	4.0414
	161.7	Ka_2 }		4.0411
	136.5	$K\beta$	(420)	4.0407
Nickel.	152.8	$K\beta$	{ 511 } { 333 }	4.0404

Results.

The three elements aluminium, copper, and silver were investigated with radiation from anticathodes of chromium, cobalt, copper, iron, and nickel.

(1) *Aluminium*.—Table I. contains the results of the measurements made on aluminium.

It will be observed that the values of the parameter vary between 4.0419 and 4.0398 and that they diminish in each case with increase of glancing angle. A further examina-

tion was carried out with this element, using very thin sheets of thickness not exceeding 0.02 mm. backed by copper. The results obtained with one of these sheets, using a copper target, are given in Table II.

The value of the parameter is now constant for each of the four lines recorded on the film. Thus the variation in the value of the parameter observed in Table I. was probably due mainly to the penetration of the X-rays in the material.

TABLE II.

Arc (<i>s</i>) in mm.	λ .	Reflecting planes.	Parameter. Å.
206.54	Ka_1 }	(422)	4.0406
203.02	Ka_2 }		4.0407
84.37	Ka_1 }	{ 511 }	4.0406
74.55	Ka_2 }	{ 333 }	4.0404

TABLE III.—Copper.

Target.	Arc (mm.).	λ .	Reflecting planes.	Parameter. Å.
Chromium.	165.4	$K\beta$	(311)	3.6081
Cobalt.	119.8	$K\beta$	(331)	3.6075
	79.5	Ka_1 }	(400)	3.6073
	70.3	Ka_2 }		3.6071
Copper.	211.6	Ka_1 }	(331)	3.6079
	208.0	Ka_2 }		3.6078
	171.6	Ka_1 }	(420)	3.6076
	167.1	Ka_2 }		3.6074
Iron.	213.4	Ka_1 }	(222)	3.6080
	210.5	Ka_2 }		3.6078
Nickel.	212.9	$K\beta$	(420)	3.6074

This effect, as will be seen below, was not so pronounced in metals of higher atomic weight, such as silver and copper. When the values of the parameter recorded in Table I. are plotted against length of arc (*s*), the values tend to a constant for small values of *s*, this constant value being about 4.0405 Å. This agrees with the value 4.0406 Å. obtained from the independent photographs taken with thin aluminium foil 0.02 mm. thick.

(2) *Copper and Silver*.—Tables III. and IV. contain the results obtained with copper and silver.

There is good agreement between the values of the parameter obtained with the different targets; the wave-lengths of the $K\alpha$ -doublet and the $K\beta$ -radiation of the materials of the targets were taken from the 'International Critical Tables'*. .

The mean values of the parameters of these three elements are therefore :—

Aluminium	$4\cdot0406 \pm 0\cdot0003 \text{ \AA.}$
Copper	$3\cdot6076 \pm 0\cdot0003 \text{ \AA.}$
Silver	$4\cdot0773 \pm 0\cdot0003 \text{ \AA.}$

TABLE IV.—Silver.

Target.	Arc (mm.).	λ .	Reflecting planes.	Parameter.
				\AA.
Cobalt.	169.2	$K\alpha_1$ }	(331)	4.0775
	165.3	$K\alpha_2$ }		4.0776
	132.7	$K\beta$	(422)	4.0771
	113.8	$K\alpha_1$ }	(420)	4.0770
	107.9	$K\alpha_2$ }		4.0771
Copper.	215.8	$K\alpha_2$	(422)	4.0776
	150.3	$K\beta$	(440)	4.0771
	112.5	$K\alpha_1$	(511)	4.0771
	105.4	$K\alpha_2$	(333)	4.0774
Iron.	198.8	$K\beta$	(331)	4.0776
	180.8	$K\alpha_1$ }	(400)	4.0771
	177.3	$K\alpha_2$ }		4.0767
	155.5	$K\beta$	(420)	4.0778
Nickel.	60.8	$K\alpha_1$ }	(422)	4.0770
	47.5	$K\alpha_2$ }		4.0769

Specimens of the spectra are reproduced in fig. 1 (Pl. XVII.) and were all taken with the larger camera, radius 69.63 mm.

Aluminium in foil and in powder form was annealed at different temperatures (250°C. and 500°C.). No difference was found in the parameter of the material due to the differences in temperature of annealing or to the form of the material.

(3) The parameters of the elements gold, iridium, lead, molybdenum, nickel, palladium, rhodium, tantalum, and tungsten were mainly measured with the smaller precision camera having an effective radius of 51.66 mm.

* Int. Crit. Tables, vol. vi. p. 36.

Radiation from a copper target only was employed in these measurements. The metals had to be annealed *in vacuo* and quenched as previously; in some cases considerable difficulty was experienced in obtaining lines well enough defined to give accurate results.

TABLE V.

Element.	Atomic number.	Arc (mm.).	λ .	Reflecting planes.	Parameter, Å.	Mean parameter.	Distance of closest approach of atoms.
Gold (f.c.).	79	80.4 75.0	Ka_1 } Ka_2 }	$\left\{ \begin{matrix} 511 \\ 333 \end{matrix} \right\}$	4.0711 4.0712	4.071 ₁	2.879
Iridium (f.c.).	77	141.3 76.7 71.0	$K\beta$ } Ka_1 } Ka_2 }	$\left\{ \begin{matrix} 511 \\ 333 \end{matrix} \right\}$ (422)	3.8312 3.8316 3.8314	3.831 ₄	2.709
Lead (f.c.).	82	151.3 148.5 73.4 67.6	Ka_1 } Ka_2 } Ka_1 } Ka_2 }	(600) (422) (620)	4.9392 4.9391 4.9392 4.9398	4.939 ₃	3.493
Molybdenum (b.c.).	42	145.7 84.6 79.5	$K\beta$ } Ka_1 } Ka_2 }	(330) (400)	3.1403 3.1402 3.1404	3.140 ₃	2.720
Nickel (f.c.).	28	127.8 124.5 88.8 83.9	Ka_1 } Ka_2 } Ka_1 } Ka_2 }	(331) (420)	3.5775 3.5773 3.5785 3.5784	3.517 ₉	2.487
Palladium (f.c.).	46	102.8 98.6	Ka_1 } Ka_2 }	(422)	3.8848 3.8853	3.885 ₁	2.717
Rhodium (f.c.).	45	51.7 42.5	Ka_1 } Ka_2 }	(422)	3.7955 3.7953	3.795 ₄	2.684
Tantalum (b.c.).	73	71.9 65.9	Ka_1 } Ka_2 }	(330)	3.3113 3.3116	3.311 ₁	2.868
Tungsten (b.c.).	74	152.5 95.7 90.9 75.3	$K\beta$ } Ka_1 } Ka_2 } $K\beta$ }	(330) (400) (420)	3.1600 3.1591 3.1588 3.1588	3.159 ₂	2.736

Table V. shows the results obtained.

These parameters may be taken to be accurate to within ± 0.001 Å. The photographs were all taken at a mean temperature of 16.5° C. Specimens of the spectra obtained are reproduced in fig. 2 (Pl. XVIII.).

The results of the measurements of the lattice parameters of all the elements so far investigated are collected in Table VI., which includes also the atomic weights and the crystal densities, calculated from the mass of the hydrogen atom (1.663×10^{-24} gm.), and the atomic weights, taken from the 36th Annual Report of the Committee on Atomic Weights published in the 'Journal of the American Chemical Society,' March 1930. The experimental values of the densities taken from the 'International Critical Tables,'

TABLE VI.

Group.	Element.	Lattice parameter (Å.).	Distance of closest approach of atoms (Å.).	Degree of purity.	Atomic weight.	Calculated crystal density at mean temp. 16.5°C . (gm. per c.c.).	Experimental density (from Int. Crit. Tables, vol. ii.) (gm. per c.c.).
		± 0.0003	Per cent.				
I.	*Ag (f.c.).	4.077 ₃	2.883	99.9	107.880	10.50 ₆	10.50 at 0°C .
	Al (f.c.).	4.040 ₆	2.857	99.6	26.97	2.69 ₃	2.703 at 20
	*Cu (f.c.).	3.607 ₆	2.551	99.9	63.57	8.93 ₇	8.94 at 20
		± 0.001					
II.	*Au (f.c.).	4.071 ₁	2.879	99.9	197.2	19.29 ₁	19.30 at 0
	*Ir (f.c.).	3.831 ₄	2.709	99.5	193.1	22.66 ₁	22.4 at 0
	*Mo (b.c.).	3.140 ₃	2.720	99.5	96.0	10.23 ₁	10.2 —
	*Ni (f.c.).	3.517 ₉	2.487	99.9	58.69	8.89 ₃	8.90 —
	*Pb (f.c.).	4.939 ₃	3.493	99.9	207.22	11.35 ₀	11.347 at 16.34
	*Pd (f.c.).	3.885 ₀	2.747	99.5	106.7	12.01 ₁	12.1 at 0
	Rh (f.c.).	3.795 ₄	2.684	99.5	102.91	12.42 ₃	12.5 at 0
	*Ta (b.c.).	3.311 ₄	2.868	99.8	181.5	16.49 ₇	16.6 —
	*W (b.c.).	3.159 ₂	2.736	99.9	184.0	19.25 ₉	19.3 —

vol. ii. p. 456, are included for comparison in the last column of the Table.

The elements marked with asterisks in Table VI. were supplied by Messrs. Johnson Matthey of Hatton Garden. We are indebted to the firm for supplying us with the information included in Column 5 of the Table as to the degree of purity of these elements. The aluminium was obtained from a single crystal of the material. The figure given in each case for the degree of purity denotes the minimum, and not the maximum purity of the material.

Bangor, October 1931.

XCVIII. *An X-Ray Investigation of the Bismuth-Antimony Alloys.* By E. G. BOWEN, M.Sc., and W. MORRIS JONES, M.A., M.Sc., F.Inst.P. (Physics Department, University College, Swansea)*.

BISMUTH and antimony are chemically related elements having similar face-centred rhombohedral crystal structures. In common with certain other related metals having similar structures (Cu and Au, for example, having face-centred cubic structures), bismuth and antimony might be expected to be completely soluble in one another.

However, no definite evidence for the existence of such an unbroken range of solid solution was obtained in the earlier metallurgical examinations of the system, while conflicting data were obtained by different observers of the electrical conductivities, thermo-electric powers, and magnetic properties of the alloys. From a consideration of all the existing data, Guertler† concluded that there was solid solution of bismuth in antimony up to 70 per cent. of bismuth, but no solution of antimony in bismuth. Alloys containing more than 70 per cent. of bismuth were considered to be a mixture of two constituents—a mixture of bismuth and the solid solution of bismuth in antimony.

More recent work by Cook‡ indicates that the rate of diffusion of antimony into bismuth is extremely small, and that by adopting a special method of preparation, alloys of every composition can be rendered homogeneous. As a result of his investigations, Cook gives the thermal diagram reproduced in fig. 1.

As with earlier workers, Cook found that microphotographs of both slowly and quickly-cooled alloys showed a mixture of two constituents, and annealing for a considerable period at a temperature just below the melting-point of bismuth did not cause the two constituents to combine. Annealing the specimens at this temperature did not allow the constituent rich in antimony to go into solution in the bismuth. But, by maintaining them at a temperature *above* the melting-point of bismuth, the bismuth constituent was kept molten, and the remaining solid constituent, containing both antimony and bismuth, was able to go completely into solution. After being subjected to this treatment, alloys which had previously been a mixture of constituents were

* Communicated by the Authors.

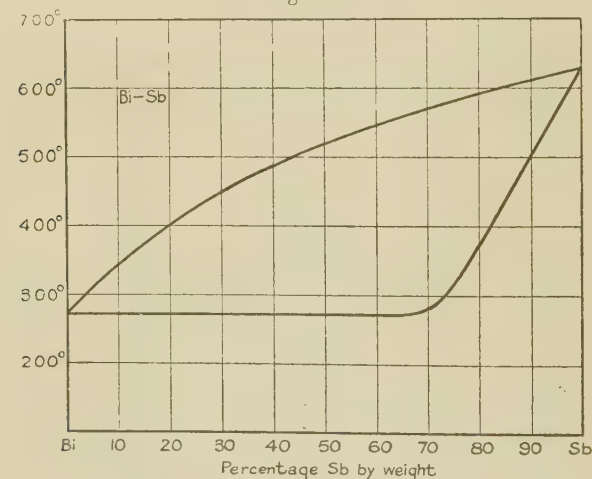
† Guertler, 'Metallographie,' i.

‡ Cook, Journ. Inst. Metals, xxviii. p. 421 (1922).

found to be completely homogeneous. The failure of previous workers to obtain homogeneous alloys over the whole range of compositions was thus due to a method of preparation which made no allowance for the extremely slow rate of diffusion of antimony into bismuth.

The present investigation was undertaken with the object of determining, by X-ray methods, whether the alloys are rendered homogeneous by the method of preparation adopted by Cook, and, if so, to find the relation between the lattice constants and the composition of the alloys.

Fig. 1.



The equilibrium diagram due to Cook.

Preparation of the Alloys.

For the purposes of the present investigation, a series of nine alloys having composition intervals of 10 per cent. were prepared and transferred immediately to a furnace standing at 280° C.—at a temperature 10° above the melting-point of bismuth. They were maintained at this temperature for 400 hours and allowed to cool gradually to room temperature over a further period of 200 hours. The alloys were allowed to cool at this very slow rate in order that the marked expansion on solidification should not set up internal strains in the specimens.

Method of Experiment.

The crystal structures of the alloys were examined by the X-ray powder method. Details of the apparatus used and

Table of Results of

Plane.	Intensity.	Pure Bi.		10 per cent. Sb.		20 per cent. Sb.		30 per cent. Sb.
		d/n .	$a_0/2$.	d/n .	$a_0/2$.	d/n .	$a_0/2$.	d/n .
100 (2)	St.	3.260	3.266	3.247	3.253	3.228	3.234	3.215
110 (2)	M	2.359	3.273	2.349	3.259	2.329	3.230	2.324
$\bar{1}10$ (2)	M.	2.260	3.266	2.255	3.260	2.241	3.242	2.228
111 (2)	W.	1.859	3.269	1.852	3.257	1.844	3.244	1.832
100 (4)	W.	1.634	3.270	1.627	3.259	1.614	3.234	1.607
331	V.W.	1.549	3.271	1.546	3.265	1.539	3.229	1.528
210 (2)	M.	1.484	3.270	1.480	3.260	1.465	3.227	1.461
$\bar{2}10$ (2)	M.	1.436	3.269	1.432	3.261	1.414	3.220	1.410
211 (2)	V.W.	1.381	3.272	1.379	3.267	1.368	3.239	1.360
$\bar{2}\bar{1}1$ (2)	M.	1.323	3.268	1.321	3.264	1.308	3.232	1.301
$\bar{2}11$ (2)	W.	1.306	3.269	1.302	3.259	1.290	3.231	1.282
110 (4)	V.W.	1.180	3.275	1.176	3.264	—	—	—
$\bar{1}10$ (4)	W.	1.131	3.269	1.128	3.262	—	—	—
100 (6)	W.	1.090	3.276	1.084	3.258	—	—	—
Lattice dimensions in Å.U.		$a_0=6.540$, $\alpha=87^\circ 34'$.		$a_0=6.520$, $\alpha=87^\circ 33'$.		$a_0=6.466$, $\alpha=87^\circ 32'$.		$a_0=6.4$, $\alpha=87^\circ$.

Explanation of the Table.—The first column gives the Miller indices of reflection (d/n) and the half-lattice edge ($a_0/2$) deduced from each line on a film. In each case the plane spacings correspond to a face-centred rhombohedral structure.

[To face p. 1031.

th-Antimony Alloys.

0 per cent. Sb.		70 per cent. Sb.		80 per cent. Sb.		90 per cent. Sb.		Pure Sb.	
<i>n</i> .	$a_0/2$.	<i>d/n</i> .	$a_0/2$.	<i>d/n</i> .	$a_0/2$.	<i>d/n</i> .	$a_0/2$.	<i>d/n</i> .	$a_0/2$.
46	3·152	3·136	3·142	3·120	3·126	3·114	3·120	3·107	3·113
86	3·169	2·263	3·138	2·250	3·119	2·248	3·116	2·246	3·113
83	3·158	2·169	3·138	2·155	3·118	2·154	3·118	2·145	3·105
95	3·158	1·775	3·138	1·769	3·113	1·768	3·112	1·766	3·109
80	3·166	1·567	3·140	1·560	3·126	1·554	3·124	1·552	3·110
04	3·171	1·490	3·142	1·482	3·125	1·479	3·118	1·475	3·110
38	3·166	1·429	3·146	1·419	3·125	1·416	3·117	1·411	3·106
88	3·162	1·330	3·144	1·368	3·118	1·368	3·118	1·364	3·109
37	3·163	1·331	3·148	1·326	3·137	1·317	3·113	1·315	3·109
79	3·161	1·271	3·141	1·263	3·122	1·262	3·120	1·259	3·112
58	3·152	—	—	—	—	1·241	3·111	1·241	3·111
39	3·158	—	—	—	—	—	—	1·122	3·110
96	3·171	—	—	1·082	3·131	—	—	1·076	3·115
50	3·156	—	—	1·041	3·129	—	—	1·034	3·108
$a_0=6·323$, $\alpha=87^\circ 28'$.		$a_0=6·283$, $\alpha=87^\circ 27'$.		$a_0=6·248$, $\alpha=87^\circ 26'$.		$a_0=6·234$, $\alpha=87^\circ 25'$.		$a_0=6·220$, $\alpha=87^\circ 24'$.	

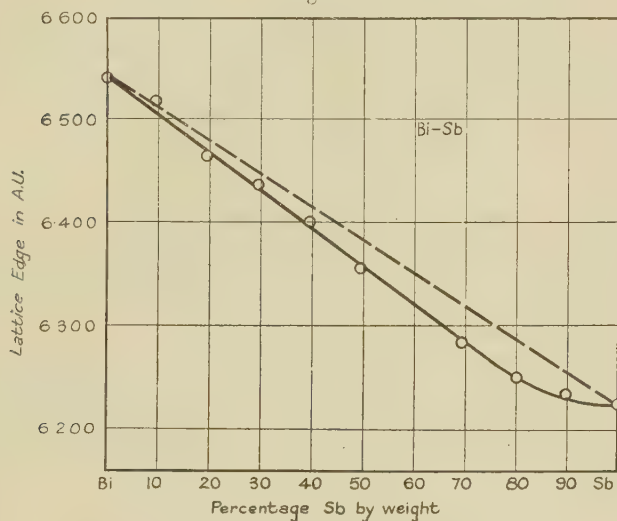
e lines on the films. Subsequent columns contain in turn the plane spacings
n of the value determined from all the lines on a film of that alloy. In every

of the precautions observed to ensure accuracy have been described in previous papers*. Adopting the procedure outlined there, the accuracy of measurement in the present investigation can be estimated as within 1 in 1000.

Results.

The complete results for all nine alloys and for bismuth and antimony are given in the Table. The first column gives the Miller indices of reflecting planes, and the second the intensity of the lines on the films. Subsequent columns contain in turn the plane spacings, deduced from measurements on films of each alloy. In every case the plane

Fig. 2.



The variation of lattice edge with composition of the bismuth-antimony alloys.

spacings correspond to a face-centred rhombohedral structure and from each observation of plane spacing is calculated a value of the lattice edge for this structure. The mean value of the lattice edge determined from all the lines on a film is taken as the lattice constant for that alloy.

The lattice constants of the pure metals are:—

Bismuth $a_0 = 6.540$ A.U., $\alpha = 87^\circ 34'$,

Antimony $a_0 = 6.220$ A.U., $\alpha = 87^\circ 24'$,

* Morris Jones and Evans, *Phil. Mag.* iv. p. 1302 (1927). Grime and Morris Jones, *Phil. Mag.* vii. p. 1113 (1929).

the angles of the rhomb differing by only 10'. If in the alloys the variation of angle with composition is linear, the change of angle from one alloy to the next would be a change of but 1', while the accuracy of measurement is only sufficient to detect a change of 4' to 5'. In the calculation of the lattice edge for each alloy it has therefore been necessary to assume that the change of angle with composition is linear. A small deviation of 2' to 3' from linearity would make no difference to the calculated lattice edge, while a deviation greater than 5' would have been evident. The results in this instance are consistent with a linear change of angle with composition.

The change of lattice edge with composition is plotted in fig. 2. The curve is almost linear; there is slight curvature only at the antimony end for alloys containing up to 25 per cent. of bismuth. This emphasises that on the introduction of the larger bismuth atom there is at first only a slight expansion of the antimony lattice, while at the bismuth end, introduction of the smaller antimony atom into the lattice of pure bismuth causes an immediate contraction of the lattice. The curve illustrates an effect which is generally exhibited in solid solutions of extended range:—the contraction of a lattice due to the introduction of smaller atoms generally takes place as anticipated, but expansion against the forces in the lattice on substitution of larger atoms does not occur so readily.

Prolonged exposure of alloys rich in bismuth gave films showing no trace of a mixture of constituents as suspected by Guertler. The regularity of the curve of the lattice edge plotted against composition is additional evidence that there is no region over which there is a mixture of constituents.

Summary.

Alloys of bismuth and antimony prepared and annealed in the usual way are a mixture of two constituents, but Cook (Jour. Inst. Metals, xxviii. p. 421, 1922) showed that they can be rendered homogeneous by maintaining them at a temperature above the melting-point of bismuth for a considerable time and allowing to cool very slowly. This is verified in the present investigation by an X-ray study of the crystal structures of a series of such alloys. Measurements of the lattice constants of the alloys show that the change of lattice edge with composition is almost linear over the whole range of compositions.

XCIX. *Operational Calculus.*—Part I. *The Definition of an Operational Representation of a Function and some Properties of the Operator derived from this Definition.*
By H. V. LOWRY*.

SUMMARY.

THE first part of this paper contains a discussion of some fundamental points in the theory of operational calculus. In particular, it is shown that there are good reasons for preferring the definition of an operational representation of a function by Bromwich's contour integral along a contour from $-\infty - 0i$ to $-\infty + 0i$ to the definition by Carson's integral.

Some fundamental properties of an operational representation $f(p)$ of a function $h(x)$ are derived from this definition of $f(p)$, the most important of which are the following :—

$$h(0) = \text{Res}_\infty \left\{ \frac{f(p)}{p} \right\}, \quad . \quad . \quad . \quad (16)$$

$$h(0) = \text{Res}_0 \left\{ \frac{f(p)}{p} \right\}, \quad . \quad . \quad . \quad (20)$$

where $\text{Res}_n\{\phi(p)\}$ denotes the value of the integral

$$\frac{1}{2\pi i} \int_{-\pi}^{\pi} \phi(p_n + re^{i\theta}) i r e^{i\theta} d\theta,$$

which is the residue of $\phi(p)$ at $p = p_n$ when $\phi(p)$ has a simple pole at this point:

$$\frac{f(p)}{p} \doteq \int_0^x h(x) dx + \text{Res}_\infty \left\{ \frac{f(p)}{p^2} \right\} \quad . \quad . \quad (23)$$

$$= \int_0^x h(x) dx + \text{Res}_0 \left\{ \frac{f(p)}{p^2} \right\} \quad . \quad . \quad (24)$$

$$-p \int_a^p \frac{f(p)}{p} dp \doteq \frac{h(x)}{x} . \quad . \quad . \quad . \quad (26)$$

$$\int_0^\infty \frac{h(x)}{x} dx = (\text{Res}_0 - \text{Res}_\infty) \left[\frac{1}{p} \int_a^p \frac{f(p)}{p} dp \right]. \quad (32)$$

$$\int_0^\infty x^r h(x) dx = (\text{Res}_0 - \text{Res}_\infty) \left[\frac{1}{p} (-D)^r \frac{f(p)}{p} \right], \quad (34)$$

where r is a positive integer and $D = \frac{d}{dp}$.

* Communicated by the Author.

If $h(x)$ has no singularities except at $x=0$, and when $x \rightarrow 0$,

$$h(x) \rightarrow Ax^m,$$

and if

$$f(p) \doteq h(x),$$

then, if m is not an integer,

$$1 - \frac{2\pi i}{e^{i2\pi m}} p h(p e^{i\pi}) \doteq \frac{f(x)}{x}, \quad . \quad . \quad . \quad (37)$$

but if m is an integer,

$$-p \log p \cdot h(-p) \doteq \frac{f(x)}{x} \quad . \quad . \quad . \quad (38)$$

Some of the above results are true only if certain conditions are satisfied. The numbers against the equations are those which they have in the paper.

1. *The Definition of the Operational Representation of a given Function.*

IN the various books and papers on the operational calculus, the operational representation $f(p)$ of a function $h(x)$ is usually defined by Carson's equation

$$f(a) = a \int_0^\infty e^{-ax} h(x) dx. \quad . \quad . \quad . \quad (1)$$

Bromwich showed that

$$h(x) = \frac{1}{2\pi i} \int_L e^{px} \frac{f(p)}{p} dp, \quad . \quad . \quad . \quad (2)$$

in which $R(x) > 0$ and L is a contour from $c - i\infty$ to $c + i\infty$, $c > 0$, which passes to the right of all the singularities of $f(p)$, is a solution of Carson's equation considered as an integral equation for $h(x)$.

If $\frac{f(p)}{p} \rightarrow 0$ as $|p| \rightarrow \infty$, then

$$h(x) = \frac{1}{2\pi i} \int_L e^{px} \frac{f(p)}{p} dp = \frac{1}{2\pi i} \int_C e^{px} \frac{f(p)}{p} dp, \quad . \quad (3)$$

where C is a contour from $-\infty - 0i$ to $-\infty + 0i$ which encloses all the singularities of $f(p)$.

It appears more reasonable to say that " $f(p)$ is an operational representation of $h(x)$ if it is any solution of the integral equation"

$$h(x) = \frac{1}{2\pi i} \int_C e^{px} \frac{f(p)}{p} dp. \quad . \quad . \quad . \quad (4)$$

Although this does not define a unique $f(p)$, we can often find a value of $f(p)$ which satisfies it by expansion in series and similar methods when $h(x)$ is a function which makes Carson's integral divergent. Briefly the chief points in favour of the definition by means of equation (4), compared with the definition by Carson's equation (1), are

- A. It enables us to find operational representations of functions like $\log x$, $J_0(x)/x$, $Y_n(x)$ etc., which make Carson's integral divergent. On this point many writers are rather misleading, for after starting with Carson's definition of $f(p)$ they find, by series and otherwise, operational representations of functions which make Carson's integral divergent. In practically all cases these operators are correct when the definition (4) is used.
- B. It enables us to find functions which are given by operators which make (2) divergent.
- C. In equation (4) x can have any value, real or complex.

D.
$$p f(p) \doteq \frac{d h(x)}{dx}, \quad . \quad . \quad . \quad (5)$$

in all cases, whereas with Carson's definition this is true only if $h(0)=0$.

The disadvantages of the definition by equation (4) are :

- E. It does not define a unique $f(p)$ for any given $h(x)$.

F.
$$\frac{1}{p} f(p) = \int_0^x h(x) dx. \quad . \quad . \quad . \quad (6)$$

only under certain conditions, although with Carson's definition (6) is true whenever $f(p)$ is defined.

Neither of these disadvantages is a very real one. Actually it is sometimes very convenient that several different operators should give the same function of x ; as for instance that

$$\frac{p}{p^2+1}, \quad -\frac{p^3}{p^2+1}, \quad \frac{p}{p^2+1} + pe^p,$$

should all be operational representations of $\sin x$.

Further we shall see that when equation (6) is not true, then $\frac{1}{p}f(p)$ is the operational representation of

$$\int_{\infty}^x h(x) dx,$$

so that with our definition of $f(p)$ the operator $\frac{1}{p}f(p)$ is of more general scope than it is with Carson's definition of $f(p)$.

In what follows we shall therefore take the definition of $f(p)$ given above in italics as the basis of our work, and following Dr. B. v. d. Pol* we shall write :

$$f(p) \doteq h(x), \quad (7)$$

or occasionally

$$f(p) \cdot 1 = h(x).$$

From the definition (4) it is easily shown that

$$f\left(\frac{h}{a}\right) \doteq h(ax), \quad (8)$$

$$p f(p) \doteq \frac{d h(x)}{dx}, \quad (9)$$

$$p^{p+a} f(p) \doteq e^{-ax} h(x), \quad (10)$$

$$e^{\lambda p} f(p) \doteq h(x + \lambda), \quad \text{for all } x \text{ and } \lambda. \quad (11)$$

These relations are similar to those given by Carson and summarized by Dr. B. v. d. Pol in the second section of his paper on the "Operational Solution of Differential Equations," but there are slight differences as, for instance, that equation (11) is true here for all values of x and λ , and not for $x > \lambda$ only.

It should, however, be noted that, if n is *not integral*,

$$p^{-n} \doteq \frac{x^n}{\Pi(n)}, \quad \text{only if } R(x) > 0, \quad (12)$$

because when $R(x) < 0$ the integral in the definition (4) is divergent.

* Dr. B. v. d. Pol, "On the Operational Solution of Differential Equations," Phil. Mag. viii. p. 861.

The Function $H(x)$.

If $\frac{f(p)}{p} \rightarrow 0$ as $|p| \rightarrow \infty$ and x is real, then

$$\frac{1}{2\pi i} \int_L e^{px} \frac{f(p)}{p} dp = \begin{cases} h(x), & x > 0, \\ 0, & x < 0. \end{cases} \quad (13)$$

Following Bromwich and Jeffreys we shall write

$$H(x) = \frac{1}{2\pi i} \int_L e^{px} \frac{f(p)}{p} dp = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \quad (14)$$

and write

$$f(p) H(x) = \frac{1}{2\pi i} \int_L e^{px} \frac{f(p)}{p} dp;$$

then by (13),

$$f(p) H(x) \doteq f(p) \cdot 1 \doteq h(x), \quad \begin{cases} x > 0, \\ x < 0, \end{cases} \quad (15)$$

When $\frac{f(p)}{p} \rightarrow 0$ as $|p| \rightarrow \infty$ it is very often useful to change over from $f(p) \cdot 1$ to $f(p) H(x)$; for example, when $f(p)$ can be expanded in a series on L but cannot be expanded in a similar series on C . It should be noted that (13), (14), (15), are true only when x is real.

The object of the first part of this paper is to consider various properties of $f(p)$ which follow from the definition (4). Many of these properties are not true if $f(p)$ is defined by Carson's integral.

2. The Values of $h(0)$ and $h(\infty)$ under Particular Circumstances.

Putting $x=0$ in equation (4) we get

$$h(0) = \frac{1}{2\pi i} \int_C \frac{f(p)}{p} dp.$$

Now C can be taken to be an infinitely large circle Γ from $-\infty - 0i$ to $-\infty + 0i$, and hence :

$$\begin{aligned} h(0) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(p)}{p} dp \\ &= \text{Res}_{\infty} \left\{ \frac{f(p)}{p} \right\}, \quad \text{say.} \quad (16) \end{aligned}$$

If $\frac{f(p)}{p}$ has a simple pole at $p = \infty$, then $\text{Res}_\infty \left\{ \frac{f(p)}{p} \right\}$ is the residue of $\frac{f(p)}{p}$ at infinity; also if $\frac{f(p)}{p} \rightarrow 0$ as $|p| \rightarrow \infty$, then,

$$\text{Res}_\infty \left\{ \frac{f(p)}{p} \right\} = 0,$$

but when $\frac{f(p)}{p}$ does not satisfy either of these conditions, then we cannot say anything in general about Res_∞ .

$h(\infty)$.

Suppose $\frac{f(p)}{p}$ has singularities p_1, p_2, \dots, p_n and that C is deformed into loops from $-\infty$ round the points p_1, p_2, \dots, p_n . The loop C_n may be considered to be made up of straight lines from $-\infty$ to p_n parallel to the real axis, and a small circle round p_n . Then,

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_n} e^{px} \frac{f(p)}{p} dp &= \frac{e^{p_n x}}{2\pi i} \int_0^\infty \frac{e^{-rx} f(p_n + re^{-i\pi}) - f(p_n + re^{i\pi})}{p_n - r} dr \\ &\quad \text{(from the straight lines)} \\ &+ \frac{e^{p_n x}}{2\pi i} \int_{-\pi}^\pi \frac{f(p_n + re^{i\theta})}{p_n + re^{i\theta}} i re^{i\theta} d\theta \quad \dots \dots \dots (17) \\ &\quad \text{(from the circle).} \end{aligned}$$

Now let $p_n = \alpha_n + i\beta_n$ where α_n is negative or zero, and let $R(x) > 0$, then

$$\begin{aligned} &\left| \frac{e^{p_n x}}{2\pi i} \int_0^\infty \frac{e^{-rx} f(p_n + re^{-i\pi}) - f(p_n + re^{i\pi})}{p_n - r} dr \right| \\ &< \frac{e^{\alpha_n x}}{2\pi} \int_0^\infty e^{-rx} M dr, \\ &= \frac{M e^{\alpha_n x}}{2\pi x}, \quad \dots \dots \dots (18) \end{aligned}$$

where M is the maximum value of

$$\left| \frac{f(p_n + re^{-i\pi}) - f(p_n + re^{i\pi})}{p_n - r} \right|,$$

between $r=0$ and ∞ . M is finite if $f(p)$ has not got another singularity with the same imaginary part as p_n , and when this is so, we can obtain a similar result by deforming C_n so that it passes round the other singularity by small semi-circles. From equation (18) it follows that, when $x \rightarrow +\infty$, the first term in equation (17) approaches zero. Further,

$$\frac{1}{2\pi i} \int_{-\pi}^{+\pi} \frac{f(p_n + re^{i\theta})}{p_n + re^{i\theta}} i r e^{i\theta} d\theta = \text{Res}_n \left\{ \frac{f(p)}{p} \right\}, \quad (19)$$

where

$$\begin{aligned} \text{Res}_n \left\{ \frac{f(p)}{p} \right\} &= 0, \quad \text{if } f(p) \rightarrow 0, \text{ as } p \rightarrow p_n \\ &= \text{residue of } \frac{f(p)}{p} \text{ at } p = p_n, \end{aligned}$$

if $\frac{f(p)}{p}$ has a simple pole at $p = p_n$.

Hence, we find that

$$\begin{aligned} h(\infty) &= \lim_{x \rightarrow +\infty} \sum \frac{1}{2\pi i} \int_{C_n} e^{px} \frac{f(p)}{p} dp \\ &= \lim_{x \rightarrow +\infty} \sum e^{p_n x} \text{Res}_n \left\{ \frac{f(p)}{p} \right\} \\ &= \text{Res}_0 \left\{ \frac{f(p)}{p} \right\}, \quad \text{if } \alpha_n > 0, \text{ except, of course, at } p=0, \\ \text{or} \quad &= \text{Res}_0 \left\{ \frac{f(p)}{p} \right\}, \quad \text{if } \alpha_n = 0 \text{ and the value of } \text{Res}_n \left\{ \frac{f(p)}{p} \right\} \text{ is zero at every other singularity } p_n. \quad (20) \end{aligned}$$

If $\alpha_n = 0$ and $\text{Res}_n \left\{ \frac{f(p)}{p} \right\} \neq 0$, then when x is a very large real number,

$$h(x) \sim \sum e^{i\beta_n x} \text{Res}_n \left\{ \frac{f(p)}{p} \right\}.$$

3. The Functions given by the Operator $\frac{f(p)}{p}$.

Since

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{2\pi i} \int_C e^{px} \frac{f(p)}{p^2} dp \right] &= \frac{1}{2\pi i} \int_C e^{px} \frac{f(p)}{p} dp \\ &= h(x); \end{aligned}$$

it follows that

$$\int_0^x h(x) dx = \frac{1}{2\pi i} \int_C e^{px} \frac{f(p)}{p^2} dp + A.$$

Therefore, when $x \rightarrow 0$, we find by equation (16) that

$$\int_0^x h(x) dx = \frac{1}{2\pi i} \int_C e^{px} \frac{f(p)}{p^2} dp - \text{Res}_\infty \left\{ \frac{f(p)}{p^2} \right\}, \quad (21)$$

and making $x \rightarrow \infty$ we get by equation (20)

$$\int_\infty^x h(x) dx = \frac{1}{2\pi i} \int_C e^{px} \frac{f(p)}{p^2} dp - \text{Res}_0 \left\{ \frac{f(p)}{p^2} \right\}, \quad (22)$$

provided the conditions imposed in (20) are satisfied.

In operational form (21) and (22) become

$$\frac{f(p)}{p} \doteq \int_0^x h(x) dx + \text{Res}_\infty \left\{ \frac{f(p)}{p^2} \right\} \quad (23)$$

$$\frac{f(p)}{p} \doteq \int_\infty^x h(x) dx + \text{Res}_0 \left\{ \frac{f(p)}{p^2} \right\} \quad (24)$$

Subtracting we get

$$\int_0^\infty h(x) dx = \text{Res}_0 \left\{ \frac{f(p)}{p^2} \right\} - \text{Res}_\infty \left\{ \frac{f(p)}{p^2} \right\}. \quad (25)$$

As examples of (23) and (24) we find from the operator

$$e^{-ax} \sin x \doteq \frac{p}{(p+a)^2 + 1},$$

$$\begin{aligned} \int_0^x e^{-ax} \sin x dx &\doteq \frac{1}{(p+a)^2 + 1} - \text{Res}_\infty \left[\frac{1}{p\{(p+a)^2 + 1\}} \right] \\ &= \frac{1}{(p+a)^2 + 1}, \end{aligned}$$

but

$$\begin{aligned} \int_\infty^x e^{-ax} \sin x dx &\doteq \frac{1}{(p+a)^2 + 1} - \text{Res}_0 \left[\frac{1}{p\{(p+a)^2 + 1\}} \right] \\ &= \frac{1}{(p+a)^2 + 1} - \frac{1}{a^2 + 1}. \end{aligned}$$

These include by subtraction

$$\int_0^\infty e^{-ax} \sin x dx = \frac{1}{a^2 + 1},$$

which is a particular case of Carson's integral.

4. The Operator for $\frac{h(x)}{x}$.

From equation (4).

$$\begin{aligned}\frac{h(x)}{x} &= \frac{1}{2\pi i} \int \frac{e^{px}}{p} \left\{ \frac{f(p)}{p} \right\} dp \\ &= \left[\frac{1}{2\pi i} \left\{ \int \frac{p f(p)}{p} dp \right\} \frac{e^{px}}{x} \right]_c \\ &\quad - \frac{1}{2\pi i} \int_c \frac{e^{px}}{p} \left\{ p \int \frac{f(p)}{p} dp \right\} dp \\ &= -\frac{1}{2\pi i} \int_c \frac{e^{px}}{p} \left\{ p \int \frac{f(p)}{p} dp \right\} dp,\end{aligned}$$

provided $R(x) > 0$.

Hence

$$-p \int_a \frac{f(p)}{p} dp \doteq \frac{h(x)}{x}, \quad R(x) > 0, \quad (26)$$

where a is any constant.

From equation (26) it follows that

$$-\int_a \frac{f(p)}{p} dp \doteq \int_0^x \frac{h(x)}{x} dx - \text{Res}_\infty \left\{ \frac{1}{p} \int_a \frac{f(p)}{p} dp \right\} \quad (27)$$

$$\doteq \int_\infty^x \frac{h(x)}{x} dx - \text{Res}_0 \left\{ \frac{1}{p} \int_a \frac{f(p)}{p} dp \right\}. \quad (28)$$

It is best to choose $a = \infty$ in (27) and $a = 0$ in (28) because Res_∞ and Res_0 are then usually zero (but this is not necessarily always so). We shall therefore write

$$-\int_\infty \frac{f(p)}{p} dp \doteq \int_0^x \frac{h(x)}{x} dx - \text{Res}_\infty \left\{ \frac{1}{p} \int_\infty \frac{f(p)}{p} dp \right\}, \quad (29)$$

and

$$-\int_0 \frac{f(p)}{p} dp \doteq \int_\infty^x \frac{h(x)}{x} dx - \text{Res}_0 \left\{ \frac{1}{p} \int_0 \frac{f(p)}{p} dp \right\}. \quad (30)$$

Subtracting these equations

$$\begin{aligned}\int_0^\infty \frac{f(p)}{p} dp &= \int_0^\infty \frac{h(x)}{x} dx \\ &\quad + \text{Res}_0 \left\{ \frac{1}{p} \int_\infty \frac{f(p)}{p} dp \right\} - \text{Res}_\infty \left\{ \frac{1}{p} \int_0 \frac{f(p)}{p} dp \right\},\end{aligned} \quad (31)$$

but a more convenient result is got by subtracting (27) and (28); this gives

$$\int_0^{\infty} \frac{h(x)}{x} dx = (\text{Res}_0 - \text{Res}_{\infty}) \left\{ \frac{1}{p} \int_a^p \frac{f(p)}{p} dp \right\}. \quad (32)$$

Equations (29), (30), and (31) are more exact forms of those given by Dr. B. v. d. Pol. Equation (32) is, I think, new. As an example of it consider the integral

$$\int_0^{\infty} \frac{e^{-cx} - J_0(bx)}{x} dx.$$

Since

$$e^{-cx} - J_0(bx) \doteq \frac{p}{p+c} - \frac{p}{\sqrt{p^2+b^2}},$$

$$\frac{1}{p} \int_a^p \frac{f(p)}{p} dp = \frac{1}{p} \{ \log(p+c) - \log(p+\sqrt{p^2+b^2}) - C \},$$

where C is a constant.

Hence

$$(\text{Res}_0 - \text{Res}_{\infty}) \left\{ \frac{1}{p} \int_a^p \frac{f(p)}{p} dp \right\}$$

$$= \log c - \log b + \log 2 = \log \left(\frac{2c}{b} \right).$$

Other integrals of this type can be evaluated in a similar way.

5. The Operator for $x^r h(x)$ and the Value of

$$\int_0^{\infty} x^r h(x), \text{ when } r \text{ is a Positive Integer.}$$

From the definition (4)

$$x h(x) = \frac{1}{2\pi i} \int_C x e^{px} \frac{f(p)}{p} dp$$

$$= \frac{1}{2\pi i} \left\{ e^{px} \frac{f(p)}{p} \right\}_C - \frac{1}{2\pi i} \int_C e^{px} D \left\{ \frac{f(p)}{p} \right\} dp,$$

where $D = \frac{d}{dp}$.

Hence, if $R(x) > 0$,

$$x h(x) = - \frac{1}{2\pi i} \int_C e^{px} D \left\{ \frac{f(p)}{p} \right\} dp$$

$$\doteq - p D \left\{ \frac{f(p)}{p} \right\},$$

and so by induction

$$x^r h(x) \doteq p(-D)^r \left\{ \frac{f(p)}{p} \right\} . \quad . \quad . \quad . \quad (33)$$

Therefore by equation (25) we get

$$\int_0^\infty x^r h(x) dx = (\text{Res}_0 - \text{Res}_\infty) \left[\frac{1}{p} (-D)^r \left\{ \frac{f(p)}{p} \right\} \right] . \quad (34)$$

6. Carson's Integral.

Since

$$e^{-ax} h(x) = e^{-ax} f(p) \\ \doteq f(p+a) \frac{p}{p+a},$$

we find, by equation (25), that

$$\int_0^\infty e^{-ax} h(x) dx = (\text{Res}_0 - \text{Res}_\infty) \left\{ \frac{f(p+a)}{p(p+a)} \right\} .$$

If $\frac{f(p)}{p} \rightarrow 0$ as $|p| \rightarrow \infty$, then $\text{Res}_\infty = 0$, and if $a > \alpha_n$ at every singularity of $\frac{f(p)}{p}$, then every singularity of $\frac{f(p+a)}{p(p+a)}$ except $p=0$, lies to the left of the imaginary axis, and this function has therefore a simple pole at $p=0$. Hence

$$\text{Res}_0 \left\{ \frac{f(p+a)}{p(p+a)} \right\} = \frac{f(a)}{a},$$

and therefore

$$\int_0^\infty e^{-ax} h(x) dx = \frac{f(a)}{a}, \quad . \quad . \quad . \quad (35)$$

provided

$$(A) \quad \frac{f(p)}{p} \rightarrow 0 \text{ as } |p| \rightarrow \infty,$$

$$(B) \quad a > \alpha_n \text{ at every singularity of } f(p).$$

From this result we conclude that there is only one value of $f(p)$ which makes

$$f(p) \doteq h(x) \quad \text{and} \quad \frac{f(p)}{p} \rightarrow a \text{ as } |p| \rightarrow \infty.$$

We may, for instance, find a value of $f(p)$ which makes

$$f(p) \doteq h(x),$$

but does not make $\frac{f(p)}{p} \rightarrow 0$ as $|p| \rightarrow \infty$. This operational representation of $h(x)$ cannot therefore be used in Carson's integral, but there must be another operator $F(p)$ which gives $h(x)$ and also satisfies the required conditions. This means, of course, that

$$f(p) - F(p) \doteq 0.$$

As an example, consider the function $J_n(x)$:

$$\frac{p}{\sqrt{p^2+1}} (\sqrt{p^2+1}-p)^n \doteq J_n(x),$$

and $\frac{f(p)}{p} \rightarrow 0$, as $|p| \rightarrow \infty$ if $n > -\frac{1}{2}$, so

$$\int_0^\infty e^{-ax} J_n(x) dx = \frac{1}{\sqrt{a^2+1}} (\sqrt{a^2+1}-a)^n, \quad n > \frac{1}{2}.$$

But, if n is a negative integer, then we can find another operator for $J_n(x)$ which satisfies the condition $\frac{f(p)}{p} \rightarrow 0$ as $|p| \rightarrow \infty$, for then

$$\begin{aligned} J_n(x) &= (-1)^n J_{-n}(x) \\ &\doteq \frac{(-1)^n p}{\sqrt{p^2+1}} (\sqrt{p^2+1}-p)^{-n}, \end{aligned}$$

and therefore

$$\int_0^\infty e^{-ax} J_n(x) dx = \frac{(-1)^n}{\sqrt{a^2+1}} (\sqrt{a^2+1}-a)^{-n},$$

where n is a negative integer. It is easily verified that in this case the difference between the two operators for $J_n(x)$, namely,

$$\frac{p}{\sqrt{p^2+1}} (\sqrt{p^2+1}-p)^n \quad \text{and} \quad \frac{p}{\sqrt{p^2+1}} (\sqrt{p^2+1}-p)^{-n},$$

is a polynomial in p and is therefore an operational representation of zero.

6. An Inversion Formula.

Suppose the function $h(x)$ has no singularities except at $x=0$, and that when $x \rightarrow 0$

$$h(x) \rightarrow A x^n, \quad n > -1.$$

Then

$$\begin{aligned}
 & \int_C e^{ap} h(pe^{-i\pi}) dp \\
 &= - \int_{\infty}^0 e^{-ax} h(x) dx - \int_0^{\infty} e^{-ax} h(xe^{2\pi i}) dx \\
 &+ \lim_{r \rightarrow 0} \int_{-\pi}^{\pi} h\{re^{i(\theta+\pi)}\} i re^{i\theta} d\theta \\
 &= \int_0^{\infty} e^{-ax} h(x) (1 - e^{i2\pi m}) dx \\
 &= (1 - e^{i2\pi m}) \left\{ \frac{f(a)}{a} \right\}, \quad . \quad . \quad . \quad . \quad . \quad (36)
 \end{aligned}$$

by equation (35), provided $f(p)$ satisfies the required conditions.

Equation (36) shows that

$$\frac{2\pi i}{1 - e^{i2\pi m}} p h(pe^{i\pi}) \doteq \frac{f(x)}{x}. \quad . \quad . \quad . \quad (37)$$

If $h(x)$ has no singularity at $x=0$, then the left-hand side has to be replaced by

$$\lim_{m \rightarrow 0} \frac{2\pi i p h(pe^{i\pi})}{1 - e^{i2\pi m}} = -p \log p h(-p),$$

and so

$$-p \log p h(-p) \doteq \frac{f(x)}{x}. \quad . \quad . \quad . \quad . \quad (38)$$

The formulæ which contain Legendre and Bessel functions as function and operator respectively, form a good example of this inversion. Thus

$$p^{\frac{1}{2}} I_{-n-\frac{1}{2}}(p) \doteq \cos x\pi \sqrt{\frac{2}{\pi}} P_n(x), \quad . \quad . \quad (39)$$

and $P_n(x)$ has no singularities except at $x=0$. Hence, by equation (37),

$$\frac{2\pi i}{1 - e^{i2\pi m}} \cos n\pi \sqrt{\frac{2}{\pi}} p P_n(pe^{i\pi}) \doteq x^{-\frac{1}{2}} I_{-n-\frac{1}{2}}(x).$$

The left-hand side simplifies to

$$-\sqrt{2\pi} \cos n\pi \cdot p P_n(p),$$

and so

$$-\sqrt{2\pi} \cos n\pi \cdot p P_n(p) = x^{-\frac{1}{2}} I_{-n-\frac{1}{2}}(x). \quad . \quad (40)$$

If in this equation we change n into $-n-1$, we get

$$\sqrt{\frac{2}{\pi}} p Q_n(p) = x^{-\frac{1}{2}} I_{n-\frac{1}{2}}(x), \quad . \quad . \quad . \quad (41)$$

since

$$Q_n(p) = \pi \cot n\pi P_{-n-\frac{1}{2}}(p).$$

Equations (39) and (41) are those numbered (45) and (63a) in Dr. B. v. d. Pol's paper, except that (39) has here been generalized so as to be true for all values of n instead of for integral values only.

As another example, from the equation

$$\sqrt{\frac{2^n}{\pi}} \Pi(n-\frac{1}{2}) \frac{p}{(p^2+1)^{n+\frac{1}{2}}} \doteq x^n J_n(x),$$

we get by equation (37)

$$\begin{aligned} \frac{2^n \Pi(n-\frac{1}{2})}{\sqrt{\pi}} \frac{1}{(x^2+1)^{n+\frac{1}{2}}} &\doteq \frac{2\pi i}{1-e^{i4\pi n}} \{p^{n+1} J_n(p) e^{i2\pi n}\} \\ &= \frac{2\pi i}{e^{-i2\pi n} - e^{i2\pi n}} p^{n+1} J_n(p) \\ &= \frac{-\pi}{\sin 2\pi n} p^{n+1} J_n(p), \end{aligned}$$

which becomes

$$-\pi \frac{1}{2} \left[J_n(p) \log p + \frac{dJ_n(p)}{dn} \right] p^{n+1},$$

where n is an integer.

7. The Solution of Differential Equations.

The solution of differential equations with constant coefficients by using operators has been dealt with in detail by Jeffreys, Pol, and others, but the method used by Pol to solve linear differential equations with variable coefficients can be considerably shortened by using equation (33), which is

$$x^r h(x) \doteq p(-D)^r \left\{ \frac{f(p)}{p} \right\}, \quad R(x) > 0.$$

The method which follows was suggested by the author in a paper in the 'Journal' of the Manchester College of Technology in 1927. It is in effect thus. Substitute for y

$$y = h(x) \doteq f(p),$$

and then apply equation (33) until no x is left in the equation. The resulting differential equation is one to find $f(p)$. Thus the equation

$$x \frac{d^2 y}{dx^2} + (1-n) \frac{dy}{dx} y = 0, \quad \dots \quad (42)$$

becomes

$$x p^2 f + (1-n) p f + f = 0.$$

By equation (33) this can be written

$$-p D(p f) + (1-n) p f + f = 0,$$

whence

$$-p^2 \frac{df}{dp} + (1-np) f = 0. \quad \dots \quad (43)$$

This is satisfied if

$$p^2 \frac{df}{dp} + (1-np) f = A p^r,$$

where r is any positive integer.

The general solution of this equation is

$$f(p) = C p^{-n} e^{-\frac{1}{p}} + A p^{-n} e^{-\frac{1}{p}} \int p^{n+r-2} e^{\frac{1}{p}} dp. \quad \dots \quad (44)$$

The first term is the operational representation of $C x^{\frac{1}{2}n} J_n(2\sqrt{x})$, which is one solution of (42), and the second term is the operational representation of another solution of equation (42). We can take r to be any number we please, but we naturally take the simplest, namely, $r=1$; this makes the second term of (44) become

$$\begin{aligned} & A p^{-n} e^{-\frac{1}{p}} \int p^{n-1} \left(1 + \frac{1}{p} + \frac{1}{2! p^2} + \dots \right) dp \\ &= A e^{-\frac{1}{p}} \left(\frac{1}{n} + \frac{1}{n-1} \cdot \frac{1}{p} + \frac{1}{n-2} \cdot \frac{1}{2! p^2} + \dots \right) \\ &= A \left(\frac{1}{n} - \frac{1}{n(n-1)p} + \frac{1}{n(n-1)(n-2)p^2} + \dots \right) \\ &= A \left(\frac{x}{n} - \frac{x^2}{n(n-1) \cdot 2!} + \frac{x^3}{n(n-1)(n-2) \cdot 3!} + \dots \right). \quad (45) \end{aligned}$$

It is easily verified that this is a solution of equation (42). Equation (43) is satisfied if we take

$$-p^2 \frac{df}{dp} + (1-np) f$$

equal to any operator which is an operational representation of zero; each operator of this kind will give us a value of $f(p)$ leading to a solution of (42) which may be a combination of $x^{\frac{1}{2}} J_n(2\sqrt{x})$ and (45).

In the same way we can find the operational solution of the differential equation

$$x \frac{d^2 y}{dx^2} + (1-n) \frac{dy}{dx} + y = \Pi(n + \frac{1}{2}) \frac{(x)^{n+\frac{1}{2}}}{\sqrt{\pi}}, \quad (46)$$

which is satisfied by $x^{\frac{1}{2}n} S_n(2\sqrt{x})$, where $S_n(x)$ is Struve's function

$$S_n(x) = \sum_{r=1}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{n+2r+1}}{\Pi(n+r+\frac{1}{2})\Pi(r+\frac{1}{2})}.$$

In just the same way as above we find that equation (46) transforms into

$$p^2 \frac{df}{dp} + (1-np)f = p^{-\frac{1}{2}-n} \sqrt{\pi},$$

of which the solution is

$$f = p^{-n} e^{-\frac{1}{p}} \int_a^p \frac{u^{-\frac{1}{2}} e^u}{\sqrt{\pi}} du,$$

where, in order to get Struve's function, the constant a has to be chosen so that when x is small

$$f(p) \doteq \frac{x^{\frac{1}{2}n+\frac{1}{2}}}{\Pi(n+\frac{1}{2})} \quad \text{nearly.}$$

This is so if we take

$$\begin{aligned} f(p) &= p^{-n} e^{-\frac{1}{p}} \int_{\infty}^p \frac{u^{-\frac{1}{2}} e^u}{\sqrt{\pi}} du \\ &= p^{-n} e^{-\frac{1}{p}} \int_0^{\sqrt{p}} \frac{2e^{-u^2}}{i\sqrt{\pi}} du \\ &= p^{-n} \frac{e^{-\frac{1}{p}}}{i} \operatorname{erf} \frac{i}{\sqrt{p}}. \end{aligned}$$

C. Electromagnetic Waves and Pulses.
*By W. E. SUMPNER, D.Sc., M.I.E.E.**

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1. THE INDEPENDENCE OF WAVES.

ONE of the oldest views about light, traceable as far back as the time of Huygens, is that light waves cross each other's path, or travel together, without interfering with each other in any way. Each wave behaves as if others do not exist. This assumption does not seem to have been ever questioned or discussed. Experimental evidence in its favour appears overwhelming. The accuracy of a message conveyed by a light ray, and imprinted on it by a far distant star, is never doubted by the spectrum analyst, or by the mathematician who founds upon it a theory about some remote universe. During recent years further evidence has been given by broadcasting. No receiving-set, however good, could possibly reproduce perfectly the programme of any one of a number of stations broadcasting simultaneously if any interference occurred between electromagnetic waves.

The bearing of this assumption upon the application of Maxwell's electromagnetic theory does not appear to have received any attention. A little consideration of the matter seems to show that the independence principle facilitates the use of Maxwell's theory, although it imposes some restraint. It calls for some reconsideration of such principles as superposition, interference, and reversibility in connexion with light, and also of the exact meaning to be attached to the Poynting flux of energy.

All seems to turn upon the meaning of the superposition of two vector fluxes \mathbf{E}_1 and \mathbf{E}_2 of the same kind. Any located physical entity which is *completely* represented by a vector \mathbf{E} must be neutralized by the addition at the same place of a similar entity represented by the vector equal and opposite to \mathbf{E} . But a physical entity may have some property represented by \mathbf{E} and another property represented by the square of \mathbf{E} . If now we superpose two such entities represented by vectors \mathbf{E}_1 and \mathbf{E}_2 , the first property will

* Communicated by the Author.

be represented by $\mathbf{E}_1 + \mathbf{E}_2$, while the second property will be represented by $\mathbf{E}_1^2 + \mathbf{E}_2^2$, not by $(\mathbf{E}_1 + \mathbf{E}_2)^2$. If \mathbf{E}_1 and \mathbf{E}_2 are equal and opposite, the first resultant *property* will vanish, but not so the second.

In a wave of light if the electric flux density is \mathbf{E} , the energy per unit volume is proportional to \mathbf{E}^2 . If two beams of light are directed along the same ray, the fluxes being \mathbf{E}_1 and \mathbf{E}_2 , the energies add, the sum being $\mathbf{E}_1^2 + \mathbf{E}_2^2$, but the fluxes do not combine to a single flux $\mathbf{E}_1 + \mathbf{E}_2$, since, if they did so, the energy would be $(\mathbf{E}_1 + \mathbf{E}_2)^2$, where $\mathbf{E}_1 + \mathbf{E}_2$ is a vector whose tensor may have any value between the sum or difference of the tensors of \mathbf{E}_1 and \mathbf{E}_2 . The energies are necessarily positive, but \mathbf{E}_1 and \mathbf{E}_2 may be of opposite sign.

In standard treatises on light nothing is said to suggest any conflict between the principle of the independence of waves and that of the interference of light. In some books the two principles are asserted in the same paragraph. Although it is not explicitly stated, the writer of the treatise appears to have in mind some such idea as follows:—The interference of light is a special case of the action of electromagnetic waves on matter. The effect may be optical, as in vision, or illumination; or chemical, as in photography; or a heat effect, as with a bolometer plate. In all cases matter intervenes. Two waves of vector strengths \mathbf{E}_1 and \mathbf{E}_2 have energies given by $\mathbf{E}_1^2 + \mathbf{E}_2^2$; yet the effect on matter may be assumed to be that due to an energy transfer-force represented by the vector sum $\mathbf{E}_1 + \mathbf{E}_2$, which will vanish if these two vectors are equal and opposite. The waves do not destroy each other any more than two batteries destroy each other if of equal and opposite electromotive forces and applied to the same circuit. The batteries simply neutralize each other's tendency to produce the electric current which forms the flux corresponding with the electromotive force. If $\mathbf{E}_1 + \mathbf{E}_2$ is not zero the flux, or rate of yield of the medium, is in general proportional to the transfer-force $\mathbf{E}_1 + \mathbf{E}_2$, so that the rate of transfer of energy is proportional to $(\mathbf{E}_1 + \mathbf{E}_2)^2$; but the energy in the wave available for transfer is not measured by $(\mathbf{E}_1 + \mathbf{E}_2)^2$, but by $\mathbf{E}_1^2 + \mathbf{E}_2^2$.

When parallel monochromatic light is passed through a grating ruled, say, with 100 lines, the waves sent along the ray corresponding with the first diffraction image of the slit contain 100 equal fluxes \mathbf{E} . The energy is $100 \mathbf{E}^2$, not $(100 \mathbf{E})^2$. Along the path of interference there are also 100 equal fluxes; but while 50 of these are $+\mathbf{E}$, the other 50 are $-\mathbf{E}$. The energy, as before, is

$$100 \mathbf{E}^2 = 50 \mathbf{E}^2 + 50 (-\mathbf{E})^2.$$

In the first case the effect on matter is determined by the transfer-force $100 E$, and the rate of transfer of energy is given by $(100 E)^2$. In the second case the effect on matter is zero. Such a view seems to contain nothing in conflict with experimental results, while it quite harmonizes the principles of independence and interference. It leads to the conclusion that the wave along the path of interference has the same energy as that along the path of reinforcement. This is not recognized as a fact, but in view of the assumption made about the action on matter it does not appear to conflict with any known experiment. Unless some arrangement of matter can be devised capable of acting in regard to light fluxes in a way similar to that of a thermionic valve in regard to the alternating electromotive forces involved in wireless telegraphy it does not seem possible that such a conclusion can experimentally be either established or disproved. The assumption about the influence of waves on matter, so far as light is concerned, is essentially what is actually made in order to explain the phenomenon of interference.

The assumption of the non-interference of radiation is made essentially in the theory of every electrical experiment, since such theory always ignores all other simultaneous electrical experiments. There is no scientific difference between two cases, one of which consists of a single experiment with two independent energy sources, and another which involves two experiments each with an independent energy source. If two separate experiments influence each other the cause is always traceable to the intervention of matter which absorbs part of the energy impinging upon it and redistributes it in other forms of radiation in addition to giving rise to reflected waves. Such phenomena are secondary and involve no direct action of waves on each other. The magnetization of light discovered by Faraday was stated by him to be a secondary effect due to the intervention of matter. The effects found by Kerr and Zeeman are other instances.

2. THE POYNTING FLUX.

In Poynting's proof* of the flux of energy there is the tacit assumption that only one energy system is involved. No one supposes that the formula applies, or was ever

* "Transfer of Energy in the Electromagnetic Field," *Phil. Trans.* 1884, pp. 343-361, or *Coll. Papers*, art. 10. A year later a second paper on a similar subject appears in *Phil. Trans.* 1885, pp. 277-306, *Coll. Papers*, art. 11.

intended to apply, to two cases, A and B, in which the system B is made to differ from that of A merely by superposing on A the magnetic field due to a permanent magnet brought up to a fixed position in A. The magnet brings an energy system with it, but does not alter the flux of energy, though it certainly alters the Poynting formula, since, if the magnetic fluxes can be combined vectorially, these fluxes are all changed, while the electric fluxes are not altered. The case taken by Poynting was of such a general character that it is possible to interpret it in many ways. He assumes that the energy per unit volume is given by one magnetic flux in conjunction with one electric flux. This is not the case in the system B, even if true of system A. Poynting dealt with two fluxes whose energies were not necessarily equal and whose directions were not necessarily perpendicular to each other. He did not consider whether a single energy system could be represented by such fluxes.

In order to bring out the points involved we give Heaviside's brief proof of Poynting's theorem, simplified by eliminating all matter and all electric or magnetic charges from the region investigated, and also by using Heaviside's compact vector methods in conjunction with his form of Maxwell's equations.

The physical quantities concerned form four pairs, electric and magnetic, which we distinguish by suffixes *e* and *m*, as follows:—

Force intensity	\mathbf{H}_e	\mathbf{H}_m ,
Flux density	\mathbf{B}_e	\mathbf{B}_m ,
Inductivity	e	m ,
Energy density	T_e	T_m .

The Maxwell equations also form four pairs:—

Law 1	$\mathbf{B}_e = e\mathbf{H}_e$; $\mathbf{B}_m = m\mathbf{H}_m$,
Law 2	$\text{div. } \mathbf{B}_e = 0$; $\text{div. } \mathbf{B}_m = 0$,
Law 3	$T_e = \frac{1}{2}\mathbf{H}_e\mathbf{B}_e$; $T_m = \frac{1}{2}\mathbf{H}_m\mathbf{B}_m$,
Law 4	$\dot{\mathbf{B}}_e = \text{curl } \mathbf{H}_m \equiv [\nabla\mathbf{H}_m]$, $-\dot{\mathbf{B}}_m = \text{curl } \mathbf{H}_e \equiv [\nabla\mathbf{H}_e]$.

We use Heaviside's units and his vector notation. A vector product is shown by square brackets and a scalar product by a superposed bar. The unit vectors parallel to the axes are *i*, *j*, *k*, with $i = [jk]$, $j = [ki]$, and $k = [ij]$. The Hamiltonian differentiator is ∇ , with components $\nabla_1, \nabla_2, \nabla_3$. A vector is printed in heavy type and its tensor in corresponding

lighter type, the components of the latter being distinguished by suffixes 1, 2, 3.

Now, if \mathbf{P} is the Poynting flux, or the vector flux of energy per unit area perpendicular to \mathbf{P} , the rate of loss of energy per unit volume is

$$\text{div. } \mathbf{P} \equiv \overline{\nabla \mathbf{P}}.$$

If the region is free from matter, and also free from electric and magnetic charges, we have

$$\begin{aligned} \overline{\nabla \mathbf{P}} &= -\frac{d}{dt} \left(\frac{1}{2e} \mathbf{B}_e^2 + \frac{1}{2m} \mathbf{B}_m^2 \right) = -\frac{1}{e} \overline{\mathbf{B}_e \dot{\mathbf{B}}_e} - \frac{1}{m} \overline{\mathbf{B}_m \dot{\mathbf{B}}_m} \\ &= -\overline{\mathbf{H}_e [\nabla \mathbf{H}_m]} + \overline{\mathbf{H}_m [\nabla \mathbf{H}_e]} \\ &= - \begin{vmatrix} E_1 & E_2 & E_3 \\ \nabla_1 & \nabla_2 & \nabla_3 \\ M_1 & M_2 & M_3 \end{vmatrix} + \begin{vmatrix} M_1 & M_2 & M_3 \\ \nabla_1 & \nabla_2 & \nabla_3 \\ E_1 & E_2 & E_3 \end{vmatrix} \\ &= \begin{vmatrix} \nabla_1 & \nabla_2 & \nabla_3 \\ E_1 & E_2 & E_3 \\ M_1 & M_2 & M_3 \end{vmatrix} = \overline{\nabla [\mathbf{H}_e \mathbf{H}_m]}, \end{aligned}$$

where

$$\mathbf{H}_e \equiv iE_1 + jE_2 + kE_3,$$

$$\mathbf{H}_m \equiv iM_1 + jM_2 + kM_3,$$

and where in working out the determinants it is supposed that the factors of any product are taken in the order of the rows from top to bottom.

We thus have

$$\overline{\nabla \mathbf{P}} = \overline{\nabla [\mathbf{H}_e \mathbf{H}_m]},$$

which means

$$\mathbf{P} = [\mathbf{H}_e \mathbf{H}_m] + \mathbf{T},$$

where \mathbf{T} is any circuital flux for which $\overline{\nabla \mathbf{T}} = 0$, and is the flux pointed out by J. J. Thomson shortly after Poynting had given his formula for the energy flux as usually stated.

Now, if we look back to laws 1, 3, and 4 and to the above proof, it will be seen that if we superpose a number of energy systems,

$$\mathbf{H}_{e1}, \mathbf{H}_{m1}; \mathbf{H}_{e2}, \mathbf{H}_{m2}; \text{etc.},$$

the energy associated with the system

$$\mathbf{H}_e = \sum_r \mathbf{H}_{er}, \quad \mathbf{H}_m = \sum_r \mathbf{H}_{mr},$$

will not correspond either in distribution or in amount with the superposed energies of the single systems. The Poynting flux will hold for each energy system alone. If we add the systems and regard them as independent we have

$$\mathbf{P} = \sum_r [\mathbf{H}_{er} \mathbf{H}_{mr}] + \mathbf{T}',$$

but this is not the same as

$$\mathbf{P} = [\mathbf{H}_e \mathbf{H}_m] + \mathbf{T};$$

the first formula represents the energy system

$$\sum_r \frac{1}{2} (e \mathbf{H}_{er}^2 + m \mathbf{H}_{mr}^2),$$

while the second represents the energy system

$$\frac{1}{2} (e \mathbf{H}_e^2 + m \mathbf{H}_m^2).$$

In Poynting's proof the second formula is assumed to hold, but we must use the first if there are two or more energy systems denoting independent disturbances. These systems must be treated as if each had a separate existence.

Oliver Heaviside discovered the energy flux independently of Poynting, but published it later. He put it in what he stated to be an *extended* form*,

$$\mathbf{W} = [(\mathbf{E} - \mathbf{e})(\mathbf{H} - \mathbf{h})].$$

It may perhaps be more correctly described as a *restricted* form, since it appears to assume that only one disturbance

* Poynting's Phil. Trans. paper dates from Jan. 1884. Heaviside's article on "Electromagnetic Induction and its Propagation" dates from Jan. 1885. The above formula for \mathbf{W} dates from Feb. 1885 (see *El. Papers*, i. p. 450). It is repeated and developed in Heaviside's Phil. Trans. paper on the "Electromagnetic Field," read June 1891, published 1892. This last paper is admitted to be obscure. To the professional mathematician the analysis itself can present no difficulty, but the basic assumptions underlying it are numerous, and are so abstract and so little recognized that no one seems to understand exactly what Heaviside means by the various forces and fluxes to which he refers, and in terms of which he expresses the behaviour of matter. In this paper a great deal of attention is given to "the proper form of the Poynting flux," pp. 440-455, after "a full and rigorous examination of all the fluxes of energy concerned." Heaviside appears to be quite clear about the need, when applying Maxwell's equations, to deal only with those forces or fluxes which correspond with the impressed forces whose action is directly under investigation. He realized that this is necessary if the energy relationships are to be correctly dealt with. He subtracted from the

can be considered at a time. Using the above notation, we can put

$$\mathbf{E} = \mathbf{H}_{e1} + \mathbf{H}_{e2} + \mathbf{H}_{e3} + \dots,$$

$$\mathbf{H} = \mathbf{H}_{m1} + \mathbf{H}_{m2} + \mathbf{H}_{m3} + \dots,$$

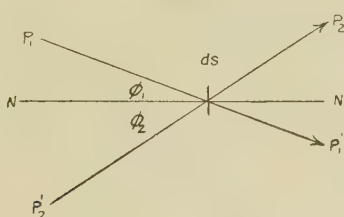
$$\mathbf{e} = \mathbf{H}_{e2} + \mathbf{H}_{e3} + \dots,$$

$$\mathbf{h} = \mathbf{H}_{m2} + \mathbf{H}_{m3} + \dots,$$

$$\mathbf{W}_1 = \mathbf{P}_1 = [\mathbf{H}_{e1} \mathbf{H}_{m1}] = [(\mathbf{E} - \mathbf{e})(\mathbf{H} - \mathbf{h})].$$

Heaviside's formula is the same as Poynting's except that it appears to insist upon the independence of disturbances. It is a mistake to add the various electric (or magnetic) fluxes to form a single flux system. The Poynting energy

Fig. 1.



fluxes, separately calculated, may for some purposes be added, yet never amalgamate to form a single flux.

The rate of outflow of energy through an element ds of a surface whose normal NN' makes angles ϕ_1 and ϕ_2 with two

total "force of the flux" any extraneous, intrinsic, or motional force, in order to get the "force of the field" with which he was directly concerned. This really amounted to applying the principle of the independence of disturbances, by using Maxwell's equations in connexion with each disturbance apart from the rest.

His remarks on Maxwell's general stress, pp. 468-9, are clear and emphatic. He distinguishes between intrinsic and induced magnetization. "Maxwell's magnetization is the sum of the two without reservation or distinction. But to unite them is against the whole behaviour of induced and intrinsic magnetization in the electromagnetic scheme of Maxwell, as I interpret it." One "keeps up a field of force," the other is "kept up by the field." "In the circuital law" they therefore "behave differently." "It is the resultant magnetization of the whole assembly that is in question, and there is a great difference between its nature according as it disappears on removal of an external cause or is intrinsic. The complete amalgamation of the two in Maxwell's formula must certainly, I think, be regarded as a false step."

Poynting fluxes \mathbf{P}_1 , \mathbf{P}_2 , directed along P_1P_1' and P_2P_2' respectively, is

$$\mathbf{P}_1(ds \cos \phi_1) + \mathbf{P}_2(ds \cos \phi_2),$$

$$\text{or} \quad ds(\mathbf{P}_1 \cos \phi_1 + \mathbf{P}_2 \cos \phi_2),$$

and this is the same as if there were a single Poynting flux,

$$\mathbf{P} = \mathbf{P}_1 \cos \phi_1 + \mathbf{P}_2 \cos \phi_2,$$

directed along NN' the normal to ds , but the vector \mathbf{P} does not represent the physical facts, since the energy flux $\mathbf{P}_1 \cos \phi_1$ goes along P_1P_1' , while the flux $\mathbf{P}_2 \cos \phi_2$ goes along P_2P_2' . None of it goes along the normal NN' .

3. WAVES AND PULSES.

Poynting, in his two papers on Maxwell's theory, 1884 and 1885, gave a number of examples illustrating the application of the Poynting flux. They, however, all involved matter and referred to states of steady flow of energy. A steady state involves fluxes which at any point are constant in time; but if the flux be regarded as a surface density of lines or tubes the individual lines are not necessarily fixed in space, since they may move sideways provided that the number threading any closed circuit remains constant in time. For present purposes we have to consider the fluxes involved in radiation. In order to do this we shall use and develop an illustration first suggested by Poynting in the 1884 paper (Coll. Papers, pp. 190-192).

The pair of circuital laws 4_c can be put in the physical or local-action form 4_v . The three laws we require then become

$$\text{Law 1} \dots \mathbf{B}_e = e\mathbf{H}_e; \quad \mathbf{B}_m = m\mathbf{H}_m,$$

$$\text{Law 3} \dots T_e = \frac{1}{2}\mathbf{H}_e\mathbf{B}_e; \quad T_m = \frac{1}{2}\mathbf{H}_m\mathbf{B}_m,$$

$$\text{Law } 4_v \dots \mathbf{H}_e = [\mathbf{B}_m\mathbf{v}_m]; \quad \mathbf{H}_m = [\mathbf{v}_e\mathbf{B}_e],$$

where \mathbf{v}_m is the velocity with which the flux \mathbf{B}_m crosses the direction of the force \mathbf{H}_e , and where \mathbf{v}_e is the velocity with which the flux \mathbf{B}_e crosses the direction of the force \mathbf{H}_m . The first of the local-action laws, 4_v , is a direct result of Faraday's discovery in 1831; the second was one which Faraday tried hard* to establish experimentally. No direct

* 'Experimental Researches,' 1658, etc. "I have long sought and still seek for an effect or condition which shall be to static electricity what magnetic force is to current electricity; for as the lines of discharge are associated with a certain transverse effect, so it appeared to me impossible but that the lines of tension or of inductive action, which of necessity precede that discharge, should also have their correspondent transverse condition or effect."

experimental evidence has yet been found for it, but it is deducible mathematically from the remote-action law of the magnetic circuit in exactly the same way as the first local-action law is derived from the remote-action law stating the electromotive force induced in a closed electric circuit. Heaviside* pointed out how extraordinary it was that Maxwell overlooked this deduction.

Now, suppose that in a region free from matter and from electric or magnetic charges we have a disturbance denoted by \mathbf{H}_e , \mathbf{H}_m , \mathbf{B}_e , and \mathbf{B}_m . We have not to consider any action at a distance, or even another disturbance in the neighbourhood. Maxwell's laws must therefore apply directly to the disturbance under consideration.

From laws 1 and 4_v used numerically it follows that, if either \mathbf{v}_m or \mathbf{v}_e is zero, all four quantities \mathbf{H} , \mathbf{B} vanish. Whatever the disturbance or its cause the disturbance must thus be in motion.

Let us assume it possible for the disturbance to move like a wave, so that, except for the motion, it remains unchanged as it progresses, and let us find what conditions are imposed by the above laws.

It follows from the assumption made that we must have

$$\mathbf{v}_m = \mathbf{v}_e = \mathbf{v}.$$

From laws 1 and 4_v, used numerically, we get

$$v^2 em = 1,$$

or the velocity must be fixed in amount. The same two laws used vectorially show that \mathbf{H}_e and \mathbf{H}_m are perpendicular to each other and also to the velocity \mathbf{v} .

If we use the value found for v in laws 1 and 3 we get numerically

$$\frac{1}{2} \mathbf{H}_e \mathbf{B}_e = \frac{1}{2} \mathbf{H}_m \mathbf{B}_m = \frac{1}{2v} \mathbf{H}_e \mathbf{H}_m.$$

Thus the electric and magnetic energies per unit volume must be equal, and there must be a fixed ratio between the two forces (or fluxes)

$$\sqrt{e} \mathbf{H}_e = \sqrt{m} \mathbf{H}_m \quad \text{and} \quad \frac{\mathbf{B}_e}{\sqrt{e}} = \frac{\mathbf{B}_m}{\sqrt{m}}.$$

The total energy per unit volume is $H_e H_m / v$, and it is moving with velocity v in a direction perpendicular to both \mathbf{H}_e and \mathbf{H}_m . The rate at which energy is flowing across a unit area

* 'Electromagnetic Theory,' i. pp. 68-69 (1893).

perpendicular to \mathbf{v} and containing the directions of \mathbf{H}_e and \mathbf{H}_m is thus the product $H_e H_m$. It is represented vectorially by the Poynting flux

$$\mathbf{P} = [\mathbf{H}_e \mathbf{H}_m].$$

Poynting did not follow up his illustration (Coll. Papers, p. 191) far enough to arrive at the Poynting flux, of which he had just given a full mathematical proof (pp. 178-181); but in his 1885 paper, after modifying Maxwell's equations and dealing with the electromagnetic field, he says (p. 217):—

“But it is, perhaps, worth noting that if we suppose that the electric intensity is produced by the motion of magnetic induction, and that the magnetic intensity is produced by the motion of electric induction, each carrying its energy with it, the right quantity of energy crosses the unit area.”

Now why the word perhaps? The significance of this result, which does not appear to have been noticed, is that unless the pulse carries all its energy with it the right quantity of energy does *not* cross the unit area, while if it does carry all its energy with it its fluxes must remain unchanged in magnitude, and no energy can be left behind the moving pulse. There can be no disturbance left in its track, since such disturbance would need energy, and there is no source of such energy.

In the above argument it has been assumed that it is possible for the disturbance to move forward unchanged. This may be admitted provided that no claim be made that the illustration yields a proof of the Poynting flux. We can, however, use the established value of the Poynting flux in conjunction with the known relationships of the fluxes in an electromagnetic wave to justify the assumption made, and thus to establish the important property of a pulse of light. Such a pulse, of duration τ , and depth $v\tau$, takes a time τ to pass any point in its path, and during that interval the corresponding energy flux has just the value needed to prove that the pulse energy per unit volume is passing on unchanged with a velocity v . The energy flux at the point is zero until the pulse reaches the point, and is also zero after the pulse has passed on. A wave of light consists of a sequence of pulses each of which behaves just as if the other pulses were absent, and as if it were quite unaffected by other waves through which it passes. This basic fact explains the result which is always assumed that the light pulses from a star convey to the spectrum analyst a correct reproduction of

the messages * given to them long ago by the star ; neither frequency nor wave-length seems to have anything to do with the process.

The examples used by Poynting in the 1884 and 1885 papers to illustrate his flux were all cases of steady flow. The only difference between such cases and those of light waves is that the stream of pulses exhibits, in regard both to magnitude and to polarization, constancy in the first case and variability in the second. If we take an energy tube bounded by four level surfaces, two of these being electric at potentials V_1 and V_2 , and the other two magnetic at potentials Ω_1 and Ω_2 , and if $d\eta$ and $d\mu$ be elements of the normals to the V and Ω surfaces respectively, we have

$$H_e = \frac{dV}{d\eta} \quad \text{and} \quad H_m = \frac{d\Omega}{d\mu},$$

and each will be perpendicular to the corresponding level surface. The Poynting flux, which is perpendicular to both, is along the energy tube, and its amount for the element of cross-area $d\eta d\mu$ is $dV \cdot d\Omega$, the integral of which across any energy tube is $(V_2 - V_1)(\Omega_2 - \Omega_1)$. Thus, even in a state of steady energy flow, the lines \mathbf{B}_e and \mathbf{B}_m are always moving, and the pulses are travelling with all their energy just as in a wave of light.

The above seems to suggest a simple way of regarding the action on the medium of an optical source. This source must exert on the medium some transfer force \mathbf{F} producing a yield ϕ whose rate $\dot{\phi}$ is such that at any moment $\mathbf{F}\dot{\phi}$ is the power transferred to the medium, while $\mathbf{F}\phi$ has the nature of energy. We may know nothing about \mathbf{F} and ϕ , yet we can always put

$$\dot{\phi} = \eta \mathbf{F} + \mathbf{G},$$

* It is not perhaps recognized that Heaviside's problem of high-speed electrical signalling was that of finding the conditions under which an electromagnetic pulse imparted to a transmitting line would afterwards behave like a pulse of light in space in spite of the interference of the matter of the transmitting line. This matter must absorb energy from the pulse and weaken the signal, but if conditions can be so arranged that the electric and magnetic energies are absorbed at equal rates the equality of the two energies will be preserved, and the effect will be only like what occurs in a light wave due to spherical spreading. No reflexions will occur to influence the effect on matter of succeeding pulses. Each pulse will go on independently and will be reduced in the same proportion, so that the whole wave, or sequence of pulses, will preserve its wave form as it progresses, and will deliver a perfect signal to the receiving station, however complex the wave form may be.

where the equation forms the definition of G , and where we can still choose η , such that for a particular interval τ_1 we have

$$\int_0^{\tau_1} F \dot{\phi} dt = \eta \int_0^{\tau_1} F^2 dt,$$

so that

$$\int_0^{\tau_1} FG dt = 0.$$

Thus FG represents a flow of energy oscillating between the source and the medium whose integral value is zero for the interval τ_1 , while ηF^2 represents a flow of energy which is permanently radiated from the source. Now, though η must depend on the action of the source during the special interval τ_1 , it does not seem unreasonable to assume that we can regard it as constant for an appreciable time, and thus independent of a small interval τ_1 . On the average FG is zero. If we multiply the equation through by F and take means for a time τ we have

$$\text{av. } F \dot{\phi} = \eta \text{ av. } F^2 + \text{av. } F \cdot G.$$

The first term on the right is the rate at which unreturning energy is radiated during the time τ , while the second represents energy which is oscillating and which must thus remain within a bounded distance measured from the source. Maxwell's theory indicates that the energy

$$\eta \tau \text{ av. } F^2 \quad \text{or} \quad \eta \int_0^{\tau} F^2 dt$$

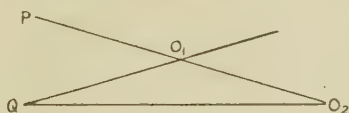
travels at a constant speed v , and at any subsequent time t is contained between two spheres of radii vt and $v(t-\tau)$. This holds however small τ may be. Thus, if the source is impulsive and exerts the transfer force F only for an infinitesimal interval of time, the corresponding radiated energy remains within an expanding spherical shell of constant thickness $v\tau$ and of expanding radius vt . This is quite in accordance with the properties of a pulse of light and with the assumption of the independence of pulses and of waves.

There seems to be no necessary relation between successive pulses in either amplitude, energy, or plane of polarization. The successive amplitudes may follow some periodic law conferring properties of frequency and wave-length, but such law is due to the source. The pulse itself can be looked upon as an independent entity, which, except for weakening due to spherical spreading, remains constant as it progresses. It is

not a wave, but the ordinate of a wave, and one whose properties cannot be represented by a simple number, since polarization as well as magnitude have to be taken into account.

Moreover, the pulse is not necessarily single. It may involve a great number of fluxes, each having its own polarization, magnitude, and associated law of sequence. Suppose P and Q are two points on a star emitting light received by a spectrum analyst at O_1 or O_2 . If PO_1 is of the order of a light year (10^{18} cm.), the surfaces of two spheres through and near O_1 , with centres at P and Q on the star, are essentially the same. A light pulse at O_1 will contain fluxes from both P and Q. These do not combine into one or even travel together, since the one travels along PO_1O_2 and the other travels along QO_1 , while the pulses from Q received at two points O_1 and O_2 have travelled by different paths. Now the star contains myriads of sources

Fig. 2.



such as P and Q, and each source may itself be complex, so that a light pulse at O_1 contains myriads of independent fluxes. This presents no theoretical difficulty if the independence of fluxes is assumed, but if the fluxes are supposed to combine into one, serious difficulties arise in connexion with the amount and distribution of the energy involved.

The general result strongly suggests that the fluxes of energy behave as if associated with some physical entity which preserves its individuality as the pulse progresses through space. Each pulse seems to consist of groups of Faraday tubes, and it is to these tubes that in all probability the individuality is due.

In space free from matter and from charges we have the relations given by law 2,

$$\overline{\nabla \mathbf{B}_e} = \text{div. } \mathbf{B}_e = 0 ; \quad \overline{\nabla \mathbf{B}_m} = \text{div. } \mathbf{B}_m = 0.$$

Now, if each flux \mathbf{B} is regarded as the density of lines per unit cross area, these relations suggest that the fluxes have individuality. They express the fact that each line has the unending characteristic associated with an ordinary line. The line goes unceasingly on, never stops at a point, never

separates into two lines, and never amalgamates with another, though it may form a closed loop. Such a line if it enters a closed region must also leave that region, and the zero divergence relation necessarily follows.

The physical aspect of Faraday tubes received much attention from Poynting *. The subject has been most fully developed by J. J. Thomson in the first chapter of 'Recent Researches' (1893), where the tubes are treated as "real physical quantities having definite sizes and shapes." The tubes "are all of the same strength." "The phenomena of electrolysis show that 'this strength' is a natural unit." It is assumed that "fractional parts of this unit do not exist." The tubes, in accordance with Faraday's views, may stretch across a vacuum. They may be open or closed, and in the latter case may be of infinitesimal size. They can "neither be created nor destroyed," so that "a change in the number passing through any fixed area must be due to the motion or deformation of the tubes."

Attention is concentrated on electrostatic tubes, since "molecular structure has an exceedingly close connexion with" them, "much closer than we have any reason to believe it has with tubes of magnetic induction." The ether is assumed to contain "multitudes of tubes" of the closed type.

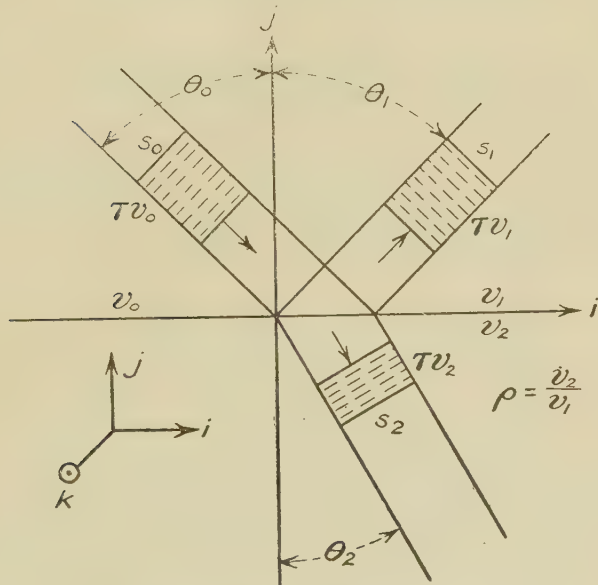
The phenomena of the magnetic field follow from the motion of the electrostatic tubes, and both Poynting and J. J. Thomson appear to regard the electrostatic tube as the primary conception; but there is so much symmetry and such an intimate connexion between the properties of electric and magnetic tubes, as shown by Maxwell's equations, that it seems as if these equations can be regarded as defining the properties of a physical concept involving both sets of tubes. Some of the above ideas about the electric tubes seem equally applicable to the magnetic ones. When an electric current produces a flux of magnetic lines through its coil these lines must presumably come from the opening out of "closed" magnetic tubes, and when the current falls to zero these lines must presumably return to the closed state.

Some conception, such as that of J. J. Thomson, seems needed to explain the physical nature of a pulse, and to turn Maxwell's theory into a physical, as distinct from a mathematical, theory of light. The properties of pulses suggest that light is to be regarded as corpuscular rather than undu-

* See his 1884 and 1885 papers, and also his paper on "Molecular Electricity," Coll. Papers, pp. 269-298.

latory. What seems wanted is the definition of some physical entity possessing at one and the same time two vector properties, the vectors being at right angles to each other and representing equal amounts of electric and magnetic energy. I have not succeeded in finding a satisfactory definition of such a quantity, but as a contribution to the subject it is proposed to show that an isolated pulse of light, when incident on the surface bounding two different non-absorbing media, splits up into two such pulses, one being reflected and the other transmitted, in accordance with the ordinary

Fig. 3.



laws of light, and with energy relations given by the usual formulæ. The surface conditions assumed in all theories of reflexion raise questions about the physical nature of the fluxes involved.

4. THE REFLEXION AND REFRACTION OF PULSES.

Let the plane of incidence ij be that of the paper, and let the positive direction \odot of the normal k to this plane be upwards, so that

$$i = [jk], \quad j = [ki], \quad k = [ij].$$

Let suffixes 0, 1, 2 distinguish corresponding quantities

referring to the incident, reflected, and transmitted rays. Let ik be the plane separating two media of inductivities e_1, m_1 and e_2, m_2 . Consider an incident pulse of duration τ within a beam of square section, the sides being unity perpendicular to the paper and also $1=s_0$ in the plane of the paper. Let θ_0, θ_1 , and θ_2 be the angles between the rays and the axis of j . If the pulse after incidence splits up into a reflected and a transmitted pulse, the depths of the pulses are $v_0\tau, v_1\tau, v_2\tau$, where v_0, v_1 , and v_2 are the velocities, while $s_0v_0\tau, s_1v_1\tau, s_2v_2\tau$ are the sections in the plane of the paper and also the pulse volumes. We assume equal magnetic inductivities $m_0=m_1=m_2$. Define a quantity ρ as

$$\rho = \frac{v_2}{v_1} = \sqrt{\frac{e_1 m_1}{e_2 m_2}} = \sqrt{\frac{e_1}{e_2}} = \sqrt{\frac{1}{k}} = \frac{1}{\mu},$$

where k is the specific inductive capacity and μ the index of refraction of the second medium as compared with that of the first.

It will be seen that the pulses exhibit the following ratios (assuming $\theta_0=\theta_1$):—

$$\begin{aligned} \text{Depths.....} & \quad v_0\tau : v_1\tau : v_2\tau = 1 : 1 : \rho, \\ \text{Cross-sections...} & \quad s_0 : s_1 : s_2 = 1 : 1 : \sigma, \\ \text{Volumes} & \quad s_0v_0 : s_1v_1 : s_2v_2 = 1 : 1 : \rho\sigma, \end{aligned}$$

where $e_1=\rho^2 e_2$, and where σ has to be defined.

Our first assumption in regard to the pulses is that the ordinary laws of reflexion and refraction hold,

$$\left. \begin{aligned} \theta_0 = \theta_1 = \phi, \quad \theta_2 = \theta, \\ \sin \theta = \rho \sin \phi, \quad \cos \theta = \sigma \cos \phi. \end{aligned} \right\} \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

The last equation defines σ , since

$$s_0 : s_1 : s_2 = \cos \theta_0 : \cos \theta_1 : \cos \theta_2.$$

If the electric *flux* is denoted by \mathbf{E} and the magnetic *flux* is denoted by \mathbf{M} , we have for equality of electric and magnetic energy per unit volume

$$\frac{E_0^2}{e_0} = \frac{M_0^2}{m_0}, \quad \frac{E_1^2}{e_1} = \frac{M_1^2}{m_1}, \quad \frac{E_2^2}{e_2} = \frac{M_2^2}{m_2}, \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

and for unaltered total energy, electric or magnetic,

$$\left. \begin{aligned} \frac{E_0^2}{e_0} &= \frac{E_1^2}{e_1} + \rho\sigma \frac{E_2^2}{e_2}, \\ \frac{M_0^2}{m_0} &= \frac{M_1^2}{m_1} + \rho\sigma \frac{M_2^2}{m_2}. \end{aligned} \right\} \cdot \cdot \cdot \cdot \cdot \quad (3)$$

The relations (3) only give one additional condition if the three relations (2) hold.

In addition, it is necessary to meet the exacting set of conditions needed to ensure, assuming the fluxes **E**, **M** in a pulse are perpendicular to each other and also to the ray, that:—

The Poynting flux for each pulse must have the correct magnitude, and must be directed along the corresponding ray in the right sense. . . . (4)

The conditions (1)–(4) are numerous, but so also are the adjustable constants needed to define the vectors. It turns out on trial that the above conditions are consistent with a solution, but are not quite sufficient to determine a single valued one.

We have not yet considered the boundary conditions to be met at the surface separating the two media. In all theories of light these are stated to be:—

The sum of the surface components of the *forces* must be equal on the two sides of the surface.

The sum of the normal components of the *fluxes* must be equal on the two sides of the surface.

These two conditions must hold both for electric and for magnetic quantities. Hence these additional conditions are four in number, and we may conveniently refer to them as the four surface conditions.

The physical meaning of these conditions is not clear in the present case. We are dealing with pulses whose properties are defined by electromagnetic equations having no reference to matter or to charges either electric or magnetic. The conditions in question are based on consideration of mechanical forces on such charges, and depend in general on the properties of matter. It turns out that in order to determine a single valued solution it is not necessary to assume all four surface conditions if conditions (1)–(4) are imposed. We reserve for the present further comment on these surface conditions.

The plane of polarization of the incident ray has yet to be defined. We shall consider two cases:

1. **E** perpendicular to the plane of incidence.
2. **M** " " " "

E perpendicular to the Plane of Incidence.

For this case we shall assume, in addition to conditions (1)-(4), the two *electric* surface conditions, but not the *magnetic* ones. We thus have

$$\frac{\mathbf{E}_0}{e_0} + \frac{\mathbf{E}_1}{e_1} = \frac{\mathbf{E}_2}{e_2},$$

or

$$\mathbf{E}_0 + \mathbf{E}_1 = \rho^2 \mathbf{E}_2.$$

There can be no component perpendicular to the surface, and each of the three fluxes **E** must be perpendicular to one of the rays. These fluxes are thus parallel, but not necessarily all of the same sign. Using the tensors algebraically we must have

$$E_0 + E_1 = \rho^2 E_2. \quad . \quad . \quad . \quad . \quad . \quad (5_e)$$

Now from (3) we have

$$E_0^2 = E_1^2 + \rho^3 \sigma E_2^2,$$

and from this and (5_e) we have

$$E_0 - E_1 = \rho \sigma E_2. \quad . \quad . \quad . \quad . \quad . \quad (6_e)$$

From (5_e) and (6_e) we thus have E_1 and E_2 in terms of E_0 .

The problem now becomes a straightforward example in vectors, the details of which may be omitted. The solution is

$$\left. \begin{aligned} \mathbf{E}_0 &= E_0 k, & \mathbf{M}_0 &= -M_0(i \cos \phi + j \sin \phi), \\ \mathbf{E}_1 &= \alpha E_0 k, & \mathbf{M}_1 &= \alpha M_0(i \cos \phi - j \sin \phi), \\ \mathbf{E}_2 &= \beta E_0 k, & \mathbf{M}_2 &= -\rho \beta M_0(i \cos \theta + j \sin \theta), \end{aligned} \right\} \quad \text{where } E_0 \text{ and } M_0 \text{ are each positive quantities related by} \quad (7_e)$$

$$\begin{aligned} E_0^2/e_0 &= M_0^2/m_0, \\ \sin \theta &= \rho \sin \phi, & \cos \theta &= \sigma \cos \phi, \\ \alpha &= \frac{\rho - \sigma}{\rho + \sigma}, & \beta &= \frac{2}{\rho(\rho + \sigma)}. \end{aligned}$$

The solution (7_e) will be found to meet completely all the conditions (1)-(4). It will also be found that the two

magnetic surface conditions are satisfied. One of these requires a relationship between the i components of the quantities \mathbf{M} , and the other a relationship between the j components of those quantities. Each relationship is satisfied, and this shows that if we had assumed the four surface conditions together with conditions (2)–(4), we could have arrived at a solution consistent with the ordinary laws of reflexion and refraction without assuming these laws, as is done by condition 1. It is easy by using a modification of Huygen's construction to see a physical reason for the laws of reflexion and refraction, but the physical meaning of the surface conditions is not clear. It is these surface conditions which in all theories of light seem to determine the ratio of the fluxes in the reflected and refracted rays. They seem to replace Newton's theory of fits in the corpuscular theory.

\mathbf{M} perpendicular to the Plane of Incidence.

In this case we assume the two magnetic surface conditions, but not the electric ones. We thus have

$$\frac{\mathbf{M}_0}{m_0} + \frac{\mathbf{M}_1}{m_1} = \frac{\mathbf{M}_2}{m_2},$$

or, using tensors algebraically as before, it follows that

$$\mathbf{M}_0 + \mathbf{M}_1 = \mathbf{M}_2; \quad \dots \quad (5_m)$$

from (3) we have

$$\mathbf{M}_0^2 - \mathbf{M}_1^2 = \rho\sigma\mathbf{M}_2^2,$$

and with the aid of (5_m) we get

$$\mathbf{M}_0 - \mathbf{M}_1 = \rho\sigma\mathbf{M}_2. \quad \dots \quad (6_m)$$

(5_m) and (6_m) give \mathbf{M}_1 and \mathbf{M}_2 in terms of \mathbf{M}_0 , and, as before, we have a determinate vector problem. The solution is

$$\left. \begin{aligned} \mathbf{E}_0 &= E_0(i \cos \phi + j \sin \phi), & \mathbf{M}_0 &= M_0 k, \\ \mathbf{E}_1 &= \alpha' E_0(i \cos \phi - j \sin \phi), & \mathbf{M}_1 &= -\alpha' M_0 k, \\ \mathbf{E}_2 &= \beta' E_0(i \cos \theta + j \sin \theta), & \mathbf{M}_2 &= \rho\beta' M_0 k, \end{aligned} \right\} \quad (7_m)$$

where E_0 and M_0 are essentially positive quantities connected by

$$E_0^2/e_0 = M_0^2/m_0,$$

and where

$$\left. \begin{aligned} \alpha' &= \frac{\rho\sigma - 1}{\rho\sigma + 1}, & \beta' &= \frac{2}{\rho(\rho\sigma + 1)}, \\ \sin \theta &= \rho \sin \phi, & \cos \theta &= \sigma \cos \phi. \end{aligned} \right\}$$

This solution will be found completely to satisfy (1)–(4) and also the two *electric* surface conditions.

The reflecting and refracting coefficients will follow from (3) in accordance with (7_e) or (7_m), whichever is appropriate to the case considered.

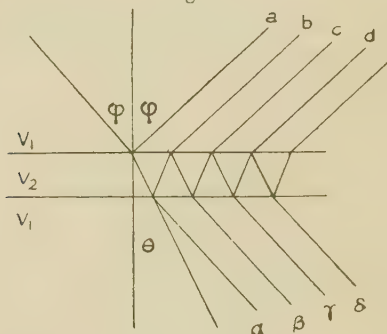
If r_{12} is the proportion of the pulse energy reflected in the first medium at the surface of the second, and if t_{12} is the corresponding ratio for the transmitted energy, we have

$$\text{for case (7}_e\text{), } r_{12} = \left(\frac{\rho - \sigma}{\rho + \sigma} \right)^2, \quad t_{12} = \frac{4\rho\sigma}{(\rho + \sigma)^2}; \quad \dots \quad (8_e)$$

$$\text{for case (7}_m\text{), } r_{12} = \left(\frac{\rho\sigma - 1}{\rho\sigma + 1} \right)^2, \quad t_{12} = \frac{4\rho\sigma}{(\rho\sigma + 1)^2} \dots \quad (8_m)$$

If we substitute for ρ and σ their values as given above in terms of ϕ and θ , the angles of incidence and refraction,

Fig. 4.



these formulæ will be found the same as those given by Fresnel's theory of light. The same formulæ were obtained on Maxwell's theory for the first time by G. F. Fitzgerald *, and later on by other workers.

It will be seen from (8_e) and (8_m) that if ρ and σ are each inverted, the values of r and t are unchanged. In any case of multiple reflexions at the surfaces of a parallel plate of a transparent medium whose velocity v_2 differs from v_1 , that of the incident beam, it will be seen from the figure that the

* G. F. Fitzgerald, Phil. Trans. pt. ii. 1880, p. 691, received Oct. 26, 1878; Shuster and Nicholson, 'Optics,' 3rd ed. 1923, pp. 238-241; Drude, 'Optics,' Eng. trans. 1902, p. 282; Wood, 'Physical Optics,' 1905, p. 294. In all cases the problem treated is that of a wave or train of pulses. An isolated constant pulse is not considered. The four surface conditions are always assumed. In spite of the great difference between the two theories of Fresnel and Maxwell, the reflecting coefficient is given by the same formula in all cases.

values of ϕ and θ are interchanged at successive transmissions, so that the values of ρ and σ are inverted. Hence

$$r_{12} = r_{21} = r, \quad t_{12} = t_{21} = t.$$

The energies of the reflected rays a, b, c, \dots will be r, rt^2, r^3t^2, \dots , while those of the transmitted rays $\alpha, \beta, \gamma, \dots$ will be $t^2, t^2r^2, t^2r^4, \dots$, expressed in each case in terms of the incident pulse energy.

The sums of these series will be found to be

$$R = \frac{2r}{1+r}, \quad T = \frac{1-r}{1+r},$$

giving

$$\text{for case } (8_e), \quad R = \frac{(\rho - \sigma)^2}{\rho^2 + \sigma^2}, \quad T = \frac{2\rho\sigma}{\rho^2 + \sigma^2}; \quad \dots \quad (9_e)$$

$$\text{for case } (8_m), \quad R = \frac{(\rho\sigma - 1)^2}{\rho^2\sigma^2 + 1}, \quad T = \frac{2\rho\sigma}{\rho^2\sigma^2 + 1}. \quad \dots \quad (9_m)$$

ρ and σ cannot be equal except for identical media. If one exceeds unity the other must be less. The product $\rho\sigma$ can be unity, in which case Brewster's law holds. For normal incidence σ is unity, and the formulæ for the e and m cases become the same.

$$\left. \begin{aligned} R &= \frac{(\rho - 1)^2}{\rho^2 + 1} = 1 - \frac{2\sqrt{k}}{k + 1} = 1 - \frac{2\mu}{\mu^2 + 1}, \\ T &= \frac{2\rho}{\rho^2 + 1} = \frac{2\sqrt{k}}{k + 1} = \frac{2\mu}{\mu^2 + 1}, \end{aligned} \right\} \quad (10)$$

where k is the specific inductive capacity and μ is the index of refraction. These last formulæ, allowing for the fact that they refer to multiple reflexions, are the same as those given by Fresnel's theory. They are, however, obtained for a single pulse, not for a complete wave. Multiple reflexion involves a sequence of pulses. These are quite separate if the time taken to traverse the plate is greater than the duration of the pulse, otherwise they overlap. In all cases they are successive, not simultaneous. For ordinary light we can assume the radiant energy is halved between pulses of types (9_e) and (9_m) , and hence a mean must be taken between the two values given for R or for T .

5. THE PHYSICAL NATURE OF PULSES.

For present purposes the primary interest of the result is that an isolated pulse behaves, when reflected or refracted,

just like a complete wave. A search for the physical meaning of the fluxes must, however, raise the question of the physical meaning of the surface conditions assumed in all theories of the reflexion of light. Fresnel's theory is entirely mechanical. Maxwell's theory does not appear to involve anything mechanical or anything about electric or magnetic charges. The surface conditions, whatever the theory, seem to be the main factor in determining the ratio of the reflected light. If the first medium e_1, m_1 represents the ether, experimental evidence shows that the second medium e_2, m_2 must contain matter. In reflexion, according to Maxwell's theory, the behaviour of matter is assumed to be summarized in the properties e_2, m_2 , and in the surface conditions; but the physical meaning of the latter has yet to be found.

The laws of reflexion and refraction follow from simple time considerations if it can be assumed that any flux in a wave front remains in that wave front. This will be evident from Huygen's construction. It also seems natural to expect that the normal components of the *fluxes* must add up to the same total on each side of the surface, though it is not at all clear, in view of the apparent individuality of fluxes, whether any flux can be split physically into two components. But what is the meaning of the surface components of the *forces* having the same total on each side of the surface? An explanation often given amounts to asserting that, unless the equality holds, a Maxwell demon can manipulate an electron so as to get work out of the system. But such a result would not conflict with the principle of the conservation of energy, since this merely asserts that in any energy transfer the energy gained in one form is lost in another. The absence of experimental evidence of such a transfer is no proof that the law of equality of surface forces must hold. The second law of thermodynamics is assumed to hold in the kinetic theory of gases, even though a sorting demon can in theory upset it.

The present investigation has merely followed up the principle of the independence of waves, the assumption of which is a very old one in connexion with light. It is not apparent that any new assumption has been made, but in order to reconcile the phenomena of interference with the independence principle it has been suggested that a compound ray of light may exist in what may be called a neutralized state, such that the ray has no observable effect on matter, although it conveys electromagnetic energy. This state occurs when the vector sum of the fluxes E is zero, a

condition necessarily involving the same relation between the corresponding fluxes \mathbf{M} .

It is not easy to suggest how experimental evidence in support of such a view can be found, and until such evidence is forthcoming the view will be looked upon as speculative. On the other hand, it can scarcely be claimed that the reflecting and refracting coefficients given by Fresnel's formulæ have as yet been thoroughly confirmed*, or that the surface conditions assumed are as yet completely established.

In the theory given above of the reflexion of an isolated pulse the ideas of frequency and phase do not arise, but it is clear that reversals of the direction of a flux do occur. If we take the case of normal reflexion, at the surface of a transparent and very thin film of a medium 2, of a ray incident in a medium 1, the \mathbf{E} flux is reversed for one of the reflected rays but not for the other, so that the dark spot in Newton's ring experiment is accounted for by the neutralization of the compound reflected ray, although the energy of each component ray must remain positive. If we consider the case of multiple reflexions nearly normal to a plate of glass, we have, from (7_e) and (8_e), with $\phi = \theta = 0$, $\sigma = 1$,

$$\begin{aligned}\mathbf{E}_0 &= E_0 k, & \mathbf{M}_0 &= -M_0 i, \\ \mathbf{E}_1 &= \alpha E_0 k, & \mathbf{M}_1 &= \alpha M_0 i, \\ \mathbf{E}_2 &= \beta E_0 k, & \mathbf{M}_2 &= -\rho \beta M_0 i,\end{aligned}$$

with

$$\begin{aligned}\alpha &= \frac{\rho - 1}{\rho + 1}, & \beta &= \frac{2}{\rho(\rho + 1)}, \\ r_{12} &= \alpha^2, & t_{12} &= 1 - \alpha^2.\end{aligned}$$

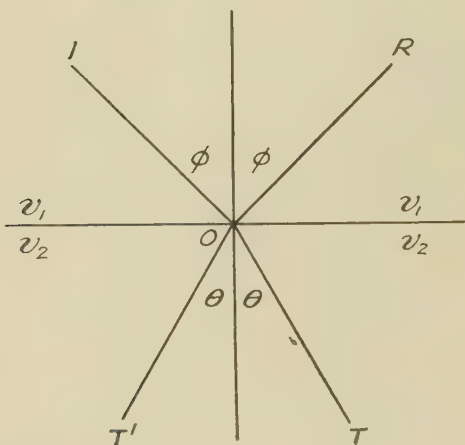
With glass ρ is less than unity for the first reflexion (see fig. 4), but for all internal reflexions it is greater than unity, since ρ is inverted. Thus the electric fluxes in the rays a, b, c, \dots are respectively $-, +, +, \dots$, as compared with that of the incident ray. The opposite signs hold for the magnetic fluxes. Thus, while the formula (9_e) holds for the reflected energy, this energy may not all be in a state to affect matter, owing to partial neutralization of the ray. This

* Lord Rayleigh [Proc. R. S. xli. pp. 275-294 (1886), or Sci. Papers, no. 138] has measured the intensity of the light reflected at "nearly perpendicular incidences." His work, as is always the case, is impressive on account of its care and skill. It also suggests that in order to confirm the surface conditions experimentally much more care and skill are needed than have probably yet been given to the matter.

effect cannot occur in the transmitted rays, since it will be seen that, whatever the sign of α , the ratios $\mathbf{E}_0 : \mathbf{E}_2$ and $\mathbf{M}_0 : \mathbf{M}_2$ are each positive.

It seemed desirable, in connexion with the present subject, to examine the "principle of reversibility" in regard to light. This was done, with the surprising result that the principle* does not appear to hold with light on the theory of Maxwell as extended by Poynting's theorem. According to this principle, if a ray of light IO is split up into a reflected ray OR and a refracted ray OT, and if these latter rays are each reversed without loss of energy, the original ray of

Fig. 5.



light is reproduced with direction reversed; but each of the reversed rays must cause a ray along OT' having energy necessarily positive, so that the compound ray sent back

* See Schuster and Nicholson, 'Optics,' 3rd ed., 1923, p. 45. The principle is based on an investigation originally due to Stokes, and is stated as follows:—"If at any time all velocities in a dynamical system are reversed and there is no dissipation of energy, the whole previous motion is reversed. Any configuration of the system which existed at a time t before reversal took place will therefore again exist at a time t after reversal." A ray of light can be reversed by reversing one only of its fluxes. If both fluxes are reversed the ray proceeds as before. A pulse of light does not appear to be a mechanical system at all. The assumption on which the principle of reversibility is based, that all velocities are reversed, does not seem to hold when a ray of light is reversed or reflected back perfectly along its original direction.

along OI cannot have the original energy. If we look at the energy coefficients (8_e) or (8_m), we have, in either case,

for energy along OI, $r_{12} \times r_{12} + t_{12} \times t_{21}$,

„ „ „ OT', $r_{12} \times t_{12} + t_{12} \times r_{21}$,

where $r_{12} = r_{21} = r$, $t_{12} = t_{21} = t$, $r + t = 1$;

the first amount is $r^2 + t^2$ and the second $2rt$. Each quantity is positive, and the sum of the two is unity. The same conclusion holds good for the reversal of the compound rays of fig. 4, as will be evident from the values of R and T given by (9_e) or (9_m).

Though the energy ratios r and t are necessarily positive, it will be found that of the two reflecting coefficients r_{12} and r_{21} one reverses the flux while the other does not. The two pulses sent along OT' have fluxes reversed, and form a neutralized ray. The two pulses sent back along OI have fluxes in the same direction.

One result which comes out is that a single energy system, represented by the incident ray IO, is turned after incidence into two energy systems, OR and OT; and that *each* of these, if reversed so as to be incident again on the bounding surface, splits up into two energy systems also, so that four energy systems result.

Our investigation points to the conclusion that an isolated pulse of light behaves in a way quite independent of the action of other pulses, and also that a pulse may itself be complex and consist of a number of distinct pulses which behave independently of each other. This in turn suggests that a single pulse may consist of a group of distinct physical entities associated in some way represented by the polarization. If such an entity exists a suitable name for it would be a *radion*. This being the object to aim at, an attempt to formulate it may be made as follows:—

For any electromagnetic pulse let

$$\lambda u = \frac{\mathbf{E}}{\sqrt{e}} = \frac{\mathbf{M}}{\sqrt{m}},$$

then $\lambda^2 u^2$ is the energy per unit volume, where λ is a number and u^2 is a unit of the nature of energy per unit volume.

This number λ represents each flux density \mathbf{E} , \mathbf{M} , in terms of units $u\sqrt{e}$ and $u\sqrt{m}$ respectively, and we may think of λ as a flux density in what we may call *lynes* per unit area in each case. Each force intensity \mathbf{E}/e and \mathbf{M}/m is also represented by λ in units which are u/\sqrt{e} and u/\sqrt{m}

respectively. The Poynting flux density $[\mathbf{EM}]/em$ is represented by

$$\lambda^2 u^2 / \sqrt{em} = v \lambda^2 u^2.$$

Thus in suitable units the number λ , which we can call the *energy number* of the pulse, represents each flux density, and also each force intensity; while in other units the number λ^2 represents the energy per unit volume, and also the Poynting flux per unit area.

Now consider a pulse of unit depth within a rectangular beam of dimensions a and b drawn in the direction of \mathbf{E} and \mathbf{M} respectively. The number of electric *lynes* will be λb and that of magnetic *lynes* will be λa . We can conceive these *lynes* as partitions perpendicular to the direction of the corresponding flux and containing the velocity direction v . These partitions will have *crossings* in $\lambda a \times \lambda b = ab\lambda^2$ lines parallel to v . The number of these *crossings* will thus be λ^2 per unit area of wave front and also per unit volume of the pulse. In the wave front the *crossings* will be the points in which the *lynes* representing the flux densities cross, but we can consider them also as lengths parallel to v within the pulse.

If d is the depth of a pulse and s its sectional area the energy is measured by $\lambda^2 sd$, and the number of *crossings* considered as points in the wave front is $\lambda^2 s$. The *crossing* considered as a line in the direction of v will be of length d , and if we consider it to consist of *radions* of fixed length put end to end the number of these *radions* will be measured by $\lambda^2 sd$, so that each *radion* will be associated with

- (i.) a fixed amount of energy, (ii.) a fixed length measured along v , and (iii.) a state of polarization defined by two vectors at right angles to each other and to v , and representing the directions of \mathbf{E} and \mathbf{M} .

The number of *radions* per unit of volume will be proportional to λ^2 , and will measure the energy of the pulse per unit volume. If we suppose these *radions* are radiating in straight lines from the source, all at constant speed v , the number per second which cross unit area of wave front will measure the Poynting flux density. We have only to suppose that each *radion* moves through space with its velocity and state of polarization unaffected by any other *radion* to account for the constant properties of the wave front due originally to the action of the source. If we consider the *lynes* as partitions perpendicular to the wave front, it will be seen that *lynes* are really level surfaces of equal potential,

magnetic or electric. If we take a pulse of depth equal to the length of a radion, the lynes divide the pulse into cells the number of which is equal to the number of crossings or to the number of radions in the pulse. These cells compare with the unit cells of Maxwell, except that instead of representing a stationary state they denote a state moving with speed v , the number of radions crossing unit area of the wave front per second being the Poynting flux density. It is tempting to imagine that the closed tube concept of J. J. Thomson may represent a closed electric induction tube linked with a closed magnetic one; that a radion consists of such a tube opened out both electrically and magnetically so as to form a right-angled cross representing the polarization of the radion, the energy needed to do this being fixed in amount and being returnable if the cross closes up again; that the unit cell of Maxwell, if electric, may contain such a tube opened out only electrically, and, if magnetic, may contain such a tube opened out only magnetically; that the unit cells of Maxwell, being fixed, and each containing half a unit of energy, may contain a stationary polarized entity which could be called a semiradion; and that the energy tubes of a beam of light formed by the lynes contain radions moving in polarized formation at speed v .

It is certainly safer, however, to regard the radion as a purely mathematical concept, in accordance with the energy known to be in a wave of light, and associating this energy with a twofold polarization and also with a velocity harmonizing with Maxwell's equations.

Dec. 5, 1931.

CL. *Notices respecting New Books.*

The Mathematical Papers of Sir William Rowan Hamilton.—Vol. I. *Geometrical Optics*. Edited for the Royal Irish Academy by A. W. CONWAY and J. L. SYNGE. Cunningham Memoir No. XIII. (Cambridge University Press, 1930. Price 50s. net.)

WHEN the scientific progress of the nineteenth century comes to be summed up in so far as it relates to mechanics and physics, the Englishman Faraday, the Scotsman Maxwell, and the Irishman Hamilton will be found to represent the three main pillars upon which the structure of modern mechanics and physics rests.

The introduction of the Characteristic Function into the treatment of Optics, and the application of its underlying idea to the study of the equations of dynamics, was an historical event comparable with the formulation of the classical laws of motion, or of the law of gravitation. Dynamics has, since the introduction of the Hamiltonian equations, been an entirely different subject, even from the subject as it left the hands of Lagrange. One has only to think of the transformation theory, integral invariants, and Hamilton's principle, with their extensive ramifications into all branches of modern physics, to realize the significance of Hamilton's great work.

As happens so often in the exact sciences, Hamilton's greatest achievement was the result of ideas which he developed as a very young man. He was born at Dublin in August 1805. At the age of seventeen, before he entered Trinity College, Dublin, as an undergraduate, he had already written and communicated to the President of the Royal Irish Academy a paper on Caustics, which contained the germ of his conception of optical propagation. The paper on Caustics was read to the Academy in December 1824, although it was not published until 1827, under the title of "The Theory of Systems of Rays," and in an entirely recast and enlarged form. This paper contains a reference to the extension of the method to dynamics, so that we can claim that, at the age of twenty-one, Hamilton had developed his characteristic function, and had realized that it could be applied to dynamics. In October 1830 Hamilton published, in a supplement to this paper, his discovery of Conical Refraction. This gained him the Cunningham Medal from the Royal Irish Academy and the Royal Medal of the Royal Society when he was still under thirty.

The present volume is the first of four volumes, which will contain the mathematical papers of Hamilton. It is issued under the auspices of the Royal Irish Academy, with the financial help of Trinity College, Dublin, University College, Dublin, and the Royal Society. The editors, Professor A. W. Conway and Professor J. L. Synge, have done their work remarkably well, and the four volumes, when issued, will constitute an indispensable part of any scientific library.

The present and first volume is entitled 'Geometrical Optics.' It consists of three parts:—Part I. gives the "Major Papers," namely, the "Theory of Systems of Rays," and three "Supplements," all published in the 'Transactions of the Royal Irish Academy,' between 1827 and 1837, and occupying nearly three hundred pages of the present volume.

Part II. is entitled "Minor Papers," published in the 'Reports of the British Association,' the 'Philosophical Magazine,' and the 'Dublin University Review.' The most interesting is the paper entitled "On a General Method of Expressing the Paths of Light, and of the Planets, by the Coefficients of a Characteristic Function," published in the 'Dublin University Review' in 1833. This gives a valuable historical review of Geometrical Optics, and

the theory of the Characteristic Function, with an indication of its application to dynamics.

While Parts I. and II. are reprints of Hamilton's published papers on Optics, Part III. consists of unpublished manuscripts. The volume of unpublished material that Hamilton left behind him at his death in 1865 was such that there are more than two hundred of his notebooks containing his meditations and the details of his researches, both successful and unsuccessful, deposited in the library of Trinity College, Dublin.

The first paper in Part III. is Hamilton's original "On Caustics" paper, submitted to the Academy in 1824. There is also included an interesting account of coma and astigmatism in the case of a symmetrical optical instrument corrected for spherical aberration. The really important work now published for the first time is one "read" to the Royal Irish Academy in 1824, but never completed. It is called "On the Improvement of the Double Achromatic Object Glass," with special application to the correction of spherical aberration and coma in an infinitely thin system.

There is an Appendix by the editors that elucidates many historical and mathematical points. Specially interesting are Notes 14 and 20. The first sets out briefly the relation of Hamilton's optical methods to dynamics, and the second shows how Schrödinger developed his Wave Mechanics from Hamilton's ideas.

This volume of Hamilton's Mathematical Papers is the first contribution to the most fitting memorial to Ireland's greatest mathematician. It is printed and produced in the characteristically efficient manner that one associates with the mathematical productions of the Cambridge University Press.

Alcoholic Fermentation. By ARTHUR HARDEN, Ph.D., D.Sc., LL.D., F.R.S. Fourth and New Edition, 1932. (London: Longmans, Green & Co. Price 15s. net.)

THE last edition of Professor Harden's well-known work was published in 1923 and comprised 157 pages of text and 31 pages of Bibliography. In the present volume the subject-matter has been extended to 198 and the Bibliography to 38 pages. The publication of this work almost coincides with the retirement of the author from the Headship of the Chemistry Department at the Lister Institute, and it is fortunate that Professor Harden has so soon found opportunity to summarize the present position of the branch of biochemistry which he has done so much to advance. The number of publications, both from his own and other laboratories, devoted to some aspect of the study of the hexose-phosphates is an enduring monument to the epoch-making observation of Harden and Young on the influence of phosphates on the fermentation of sugar by yeast-juice, which was communicated to the Chemical Society on June 1st, 1905. Although the

importance of this discovery was at once recognized, it would have been difficult to foresee at that time the wide-reaching biochemical significance to which it has since attained.

An illuminating aspect of modern work in biochemistry is the frequency with which a mechanism established for one type of metabolic change is later found to bear closely on some biochemical process which at first appears unconnected.

The author (on p. 17) writes:—"The mechanism for the decomposition of the sugar molecule is not confined to the yeast-cell, for it has been found that the same processes are involved in the conversion of carbohydrate into lactic acid in muscle, and in the decomposition of carbohydrates by bacteria, moulds, and the higher plants." And again (on p. 75): "Phosphate plays an essential part in the decomposition of carbohydrates both in the yeast-cell and in the animal organism." The high degree of specialization which, however, characterizes many biochemical processes is illustrated by recent work on the part played by pyrophosphoric acid in the chemical changes occurring in yeast and in muscle.

The use of the enzyme known as bone phosphatase is mentioned on several occasions. The value of this book to the general biochemist would have been, if possible, enhanced by a short account of the work of Robinson which led to the detection of this enzyme and to the determination of the part it plays in ossification in animals. This most valuable outcome of Harden and Young's work could hardly have been foreseen at the time of their first communication in 1905.

In the last edition the views of Neuberg on the mechanism of alcoholic fermentation and the results obtained on addition of foreign substances ("traps") to the fermenting medium were fully described. His striking experiments were, however, somewhat lacking in completeness, since methylglyoxal had at that time never been isolated from fermenting solutions, while the recognition of pyruvic acid rested on experiments the interpretation of which was in dispute. Further evidence of the participation of both methylglyoxal and pyruvic acid in alcoholic fermentation has now been obtained by Neuberg. The fermentation of magnesium hexose phosphate by sterile dried yeast preparations in which the activity of the co-enzyme has been impaired by treatment with toluene, bromo-benzene, ether, etc. can be partially arrested at either the methylglyoxal or pyruvic acid-glycerol stages, and these intermediate compounds isolated as crystalline derivatives. Glycerol and pyruvic acid have now been obtained from glucose by living yeast also. This work is summarized in the new edition and forms an interesting extension of another fundamental research of Harden and Young—that on the co-enzyme of yeast-juice.

One of the most important advances recorded in the new edition is the recognition of the hexose diphosphate of yeast fermentation as a 1:6-diphosphate of fructofuranose. A hexosemonophosphate also formed in the alcoholic fermentation of sugar is probably a

6-phosphoric ester of an aldopyranose. Chapter III. contains a valuable summary of this and other work on the constitution of the various hexose-phosphates. Recent studies of the effect of sodium fluoride, iodoacetic and bromoacetic acids on fermentation, and on formation of lactic acid by muscle, which appear to open up a wide field of work, are also summarized.

The Atomic and Molecular Forces of Chemical and Physical Interaction in Liquids and Gases, and their Effects. By R. D. KLEEMAN, B.Sc., B.A., D.Sc. [Pp. vi+133.] (London: Taylor & Francis. Price 10s. 6d. net.)

THE extensive researches of Dr. Kleeman in this subject are well known from his many contributions to scientific journals. In this book he has attempted to weld together his results into a connected whole. The problem is attacked from a classical thermodynamic standpoint. Functional properties of the law of force between atoms and molecules in gases and liquids are obtained, and are used to deduce relationships for and between various properties, such as latent heat, specific heat, viscosity, surface energy, coefficient of expansion, critical quantities, etc.

In view of the complexity of the problems considerable freedom must admittedly be allowed in assumption and approximation. While Dr. Kleeman makes ample use of this freedom, the experimental evidence he adduces in support of the relationships deduced shows that, even if they are regarded as more or less empirical, they are bound to have at least a pragmatic importance for the physical chemist.

The striking way in which it is shown how the sum of the square roots of the atomic weights of the atoms in a molecule is related to so many physical properties cannot fail to excite interest and to give food for thought to the theoretical physicist.

The arrangement of the book makes the absence of an index more than usually inconvenient.

The Combination of Observations. By DAVID BRUNT, M.A., B.Sc. Second Edition. [Pp. 239.] (Cambridge University Press, 1931.)

THIS is a useful text-book dealing in the main with the method of Least Squares, but it contains also a few other statistical problems. The first part deals with the method of Least Squares: it contains a chapter on Errors of Observation, the Gaussian Error Law, the case of one and more than one Unknown, the Weight of Observations, the problem of the Most Probable Values of the Unknown, the case of Conditioned Observations and the Rejection of Observations. The other chapters deal with alternatives to the Normal Error Law (method of moments, Pearson's types and Charlier's formula), Correlation (including the Theory of Contingency and of Partial Correlation), Harmonic Analysis (Fourier Series), and Periodogram Analysis.

CII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

February 24th, 1932.—Sir Thomas Henry Holland, K.C.S.I.,
K.C.I.E., F.R.S., President, in the Chair.

THE following communication was read :—

‘ The Geology of the N’Changa District, Northern Rhodesia.’
By Gerald Christopher Arden Jackson, D.Sc. M.A.Sc. D.I.C.
F.G.S. (Read, in the absence of the Author, by Prof. C. G.
Cullis, D.Sc. F.G.S.)

The paper treated of the general geology, petrology, correlation, and structure of the N’Changa district, Northern Rhodesia.

The area, which forms part of the original Rhodesian Congo Border Concession, covers approximately 800 square miles, and lies immediately to the south of the Belgian Congo border. It is a wooded peneplain, underlain by four principal series of ancient metamorphosed and unfossiliferous sediments: namely, Basement Schists, Muva Series, Bwana M’Kubwa Series, and Kundelungu Series. Into these have been intruded batholiths and stocks of grey and red granites and adamellites, and sill-like masses of scapolitized and uralitized gabbros and norites. The Basement Schists and Muva Series have, in addition, been invaded by large granitic intrusions of pre-Bwana M’Kubwa age: namely, the Muliashi porphyritic granite-gneiss, and possibly the M’Kushi granite-gneiss.

The complex ranges from the Archæan to possibly the Lower Palæozoic in age. The shaly and arkosic members of the Lower Bwana M’Kubwa Series and its equivalents contain the enormous replacement copper deposits occurring in Northern Rhodesia. The copper mineralization is attributed to late-stage pneumatolytic action which accompanied the intrusion of the younger grey and red ‘granites’. Details of the ores and related economic problems of the N’Changa mineralized zone are treated by the Author in a separate paper.

Petrological descriptions were given of all the principal rock types, and their distribution and correlation were discussed. Many interesting rocks were described, including a suite of gabbros and norites exhibiting progressive stages of auto-scapolitization.

*The Editors do not hold themselves responsible for the
views expressed by their correspondents.*

FIG. 2.

